Using Interval Arithmetic to Evaluate Tolerances in Reconfigurable Linear Antenna Arrays

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Contents

1 Symmetric Amplitude Tolerances - $w\left\{\mathbf{A}_n^C\right\} = 1.0$

1.1 Common Elements $P = 2$

Array Parameters:

- Number Elements: $N = 20$
- Services: Sum / Difference Beams
- Number of Common Elements: $P = 2$
- Element Spacing: $\lambda/2$

Constraints:

- Main Sum Lobe Width: $BW^{\Sigma} = 0.24u$
- Main Difference Lobe Width: $BW^{\Delta} = 0.38u$

Simulation Parameters:

- Sample Points: 2001
- Max Function Evaluations: 6000
- Max Iterations Number: 1000
- Function Tolerance: 1.0×10^{-8}
- Constraint Tolerance: 1×10^{-8}

Algorithm Behaviour:

- Simulation Time Pattern: 20 sec.

In the following figures are reported, for each iteration, the max values evaluated by the objective function and by the constraint function for the sum and difference pattern synthesis.

Sum Beam:

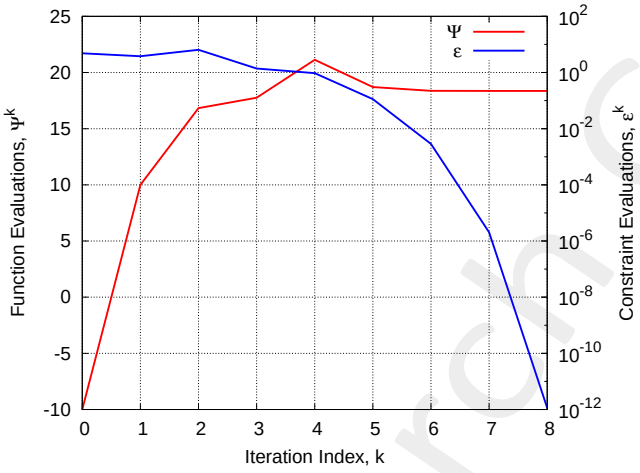

Figure 1. Sum Beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
18.36	1.1×10^{-12}

Table 1. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Difference Beam:

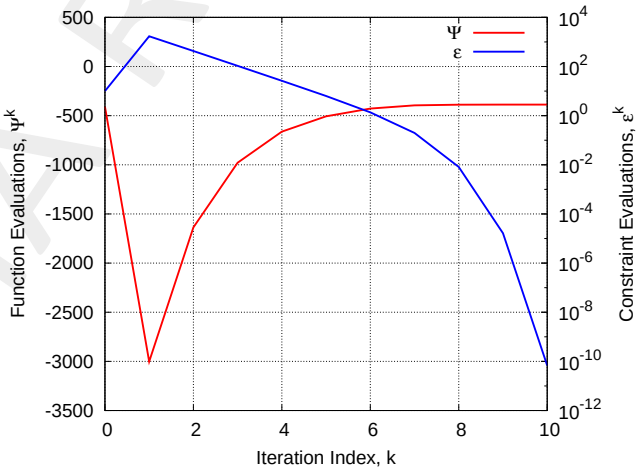

Figure 2. Difference beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
-387.9	6.8×10^{-11}

Table 2. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Excitations:

- Number of Common Elements: $P = 2$

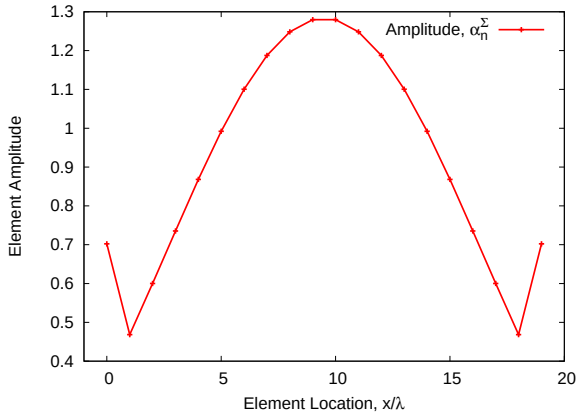

Figure 3. Sum beam's excitations amplitudes

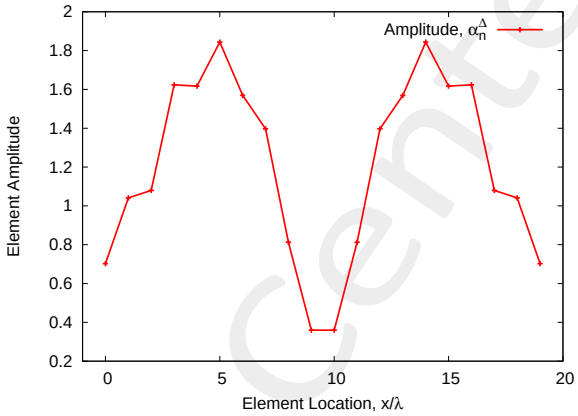

Figure 4. Difference beam's excitations amplitudes

Normalized Excitations:

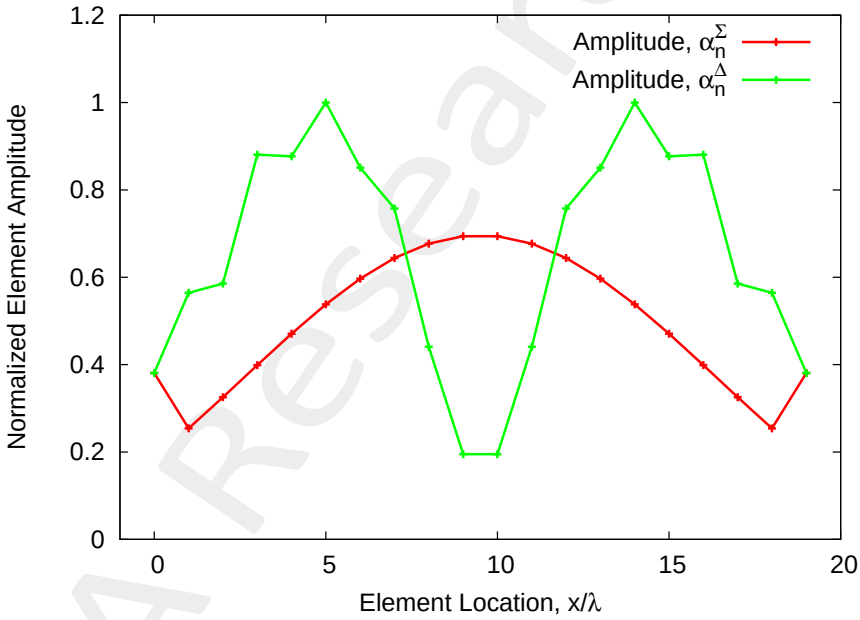

Figure 5. Sum and Difference beam's normalized excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
α_n^{Σ}	0.3808	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
α_n^{Δ}	0.3808	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195

Table 3. Sum and Difference beam's nominal amplitudes values (symmetric excitations)

Sum Pattern:

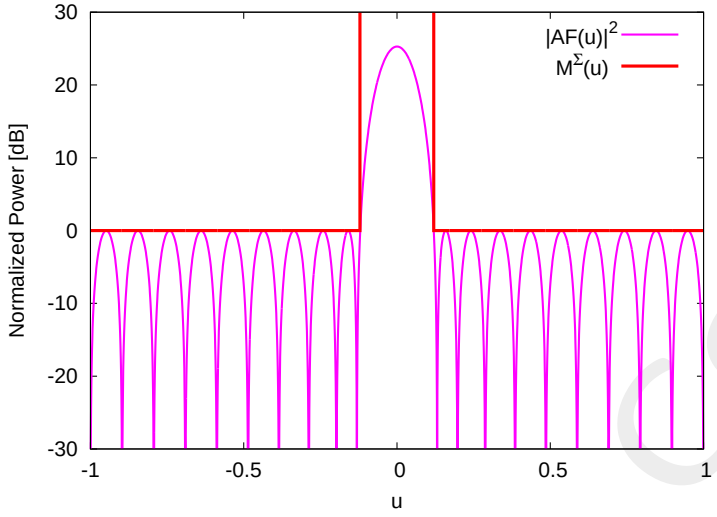

Figure 6. Synthesized Sum Pattern

SLL^{nom} [dB]	$D_{\max}^{nom}$ [dB]	BW^{nom} [u]	ψ_1 [u]
-25.28	12.65	0.104	0.131

Table 4. Sum beam's features values

Difference Pattern:

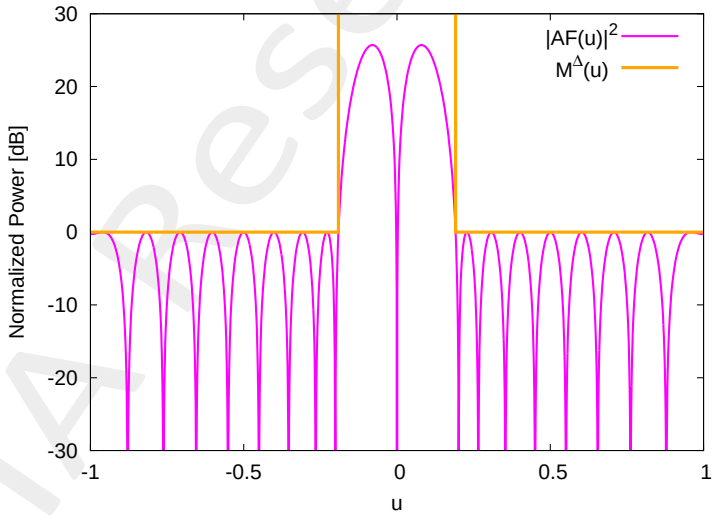

Figure 7. Synthesized Difference Pattern

SLL^{nom} [dB]	K^{nom} [1/rad]	BW^{nom} [u]	ψ_1 [u]
-25.7	1.1281	0.087	0.201

Table 5. Difference beam's features values

Tolerance Over Common Elements

- Total Tolerance: $T = 1.7617$

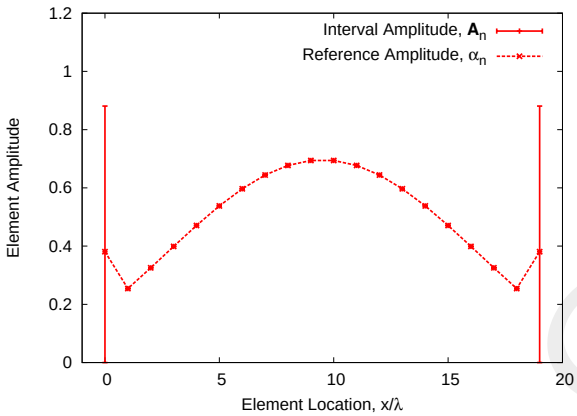

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.3808	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
$\inf\{\mathbf{A}_n\}$	0.0	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
$\sup\{\mathbf{A}_n\}$	0.8808	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
$w\{\mathbf{A}_n\}$	0.8808	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

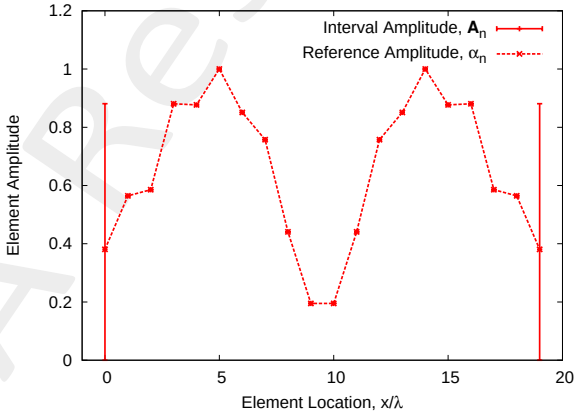

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.3808	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195
$\inf\{\mathbf{A}_n\}$	0.0	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195
$\sup\{\mathbf{A}_n\}$	0.8808	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195
$w\{\mathbf{A}_n\}$	0.8808	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

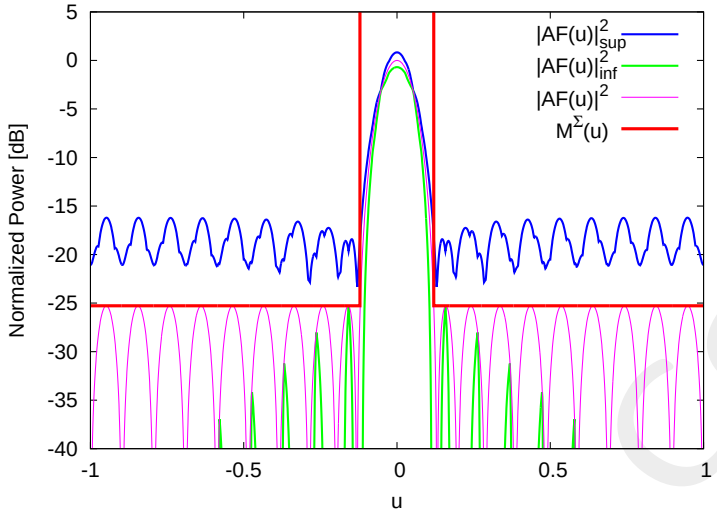

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.061	0.301
inf	-26.17	0.076	11.12	-0.69		
sup	-15.51	0.122	13.66	0.83		

Table 8. Sum Pattern Features

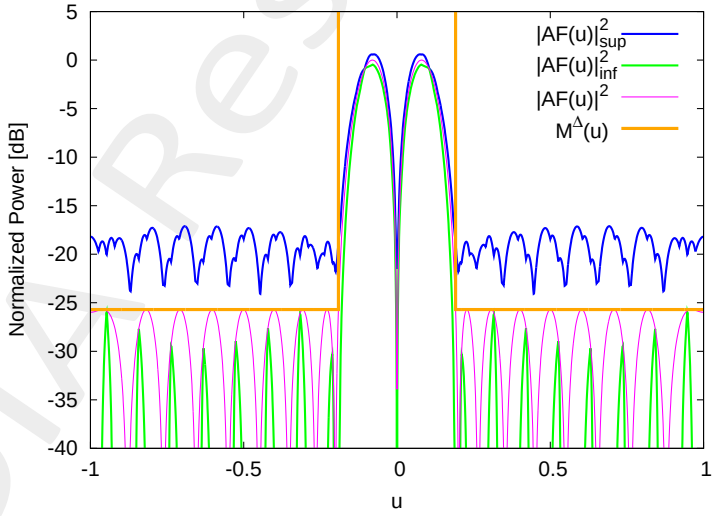

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.7	0.087	1.1281	0.0	0.0772	0.2703
inf	-26.56	0.069	0.9469	-0.46		
sup	-16.65	0.103	1.3077	0.59		

Table 9. Difference Pattern Features

Tolerance Over Not-Common Elements

- Total Tolerance: $T = 1.7.617$

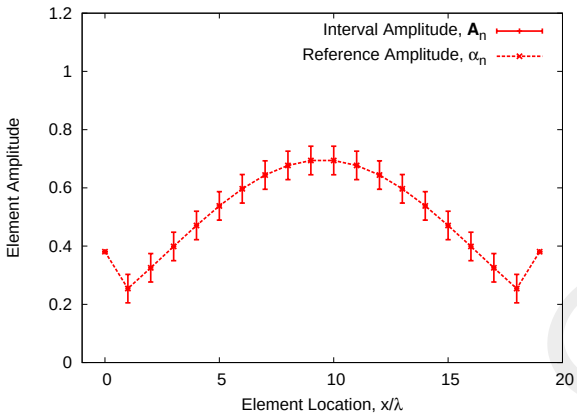

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.3808	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
$\inf\{\mathbf{A}_n\}$	0.3808	0.2051	0.2765	0.3499	0.422	0.4891	0.5479	0.5951	0.6281	0.6451
$\sup\{\mathbf{A}_n\}$	0.3808	0.303	0.3743	0.4478	0.5198	0.587	0.6458	0.693	0.726	0.743
$w\{\mathbf{A}_n\}$	0.0	0.9787	0.9787	0.9787	0.9787	0.9787	0.9787	0.9787	0.9787	0.9787

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

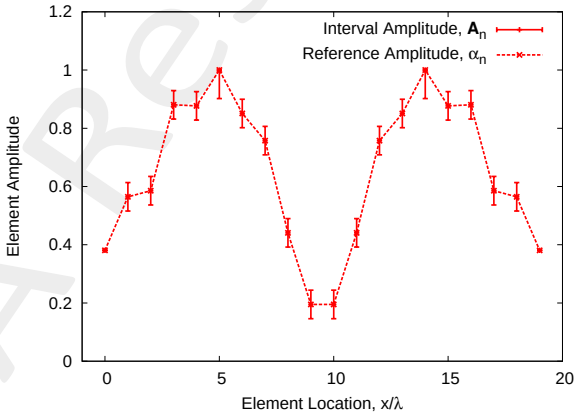

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.3808	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195
$\inf\{\mathbf{A}_n\}$	0.3808	0.5157	0.5367	0.8317	0.8281	0.9021	0.802	0.7087	0.3919	0.146
$\sup\{\mathbf{A}_n\}$	0.3808	0.6135	0.6345	0.9296	0.9259	1.0	0.8998	0.8065	0.4898	0.2439
$w\{\mathbf{A}_n\}$	0.0	0.9787	0.9787	0.9787	0.9787	0.9787	0.9787	0.9787	0.9787	0.9787

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

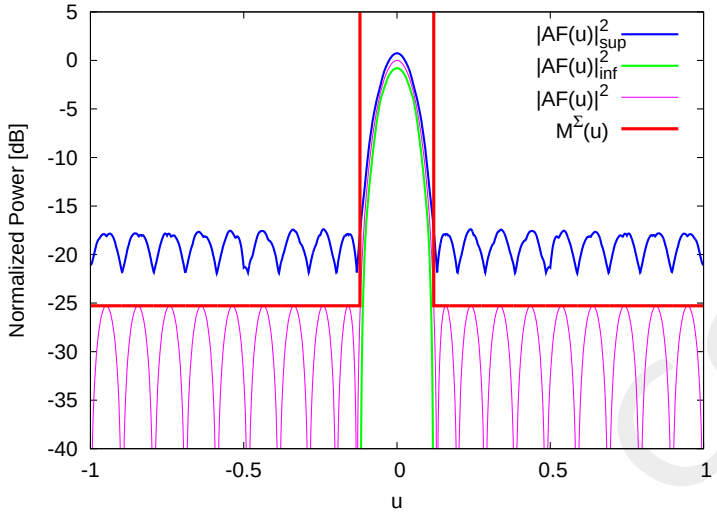

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.066	0.329
inf	-52.95	0.072	11.24	-0.8		
sup	-16.58	0.13	14.07	0.74		

Table 8. Sum Pattern Features

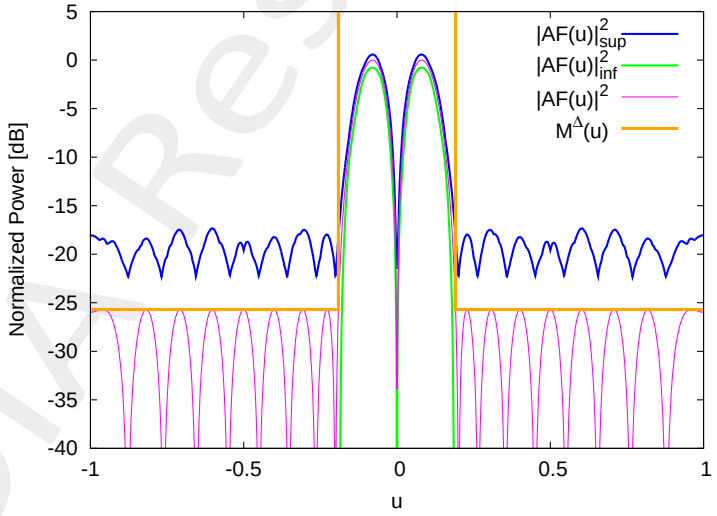

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.7	0.087	1.1281	0.0	0.0813	0.2846
inf	-48.53	0.067	1.0016	-0.77		
sup	-16.56	0.107	1.2786	0.57		

Table 9. Difference Pattern Features

1.2 Common Elements $P = 4$

Array Parameters:

- Number Elements: $N = 20$
- Services: Sum / Difference Beams
- Number of Common Elements: $P = 4$
- Element Spacing: $\lambda/2$

Constraints:

- Main Sum Lobe Width: $BW^{\Sigma} = 0.24u$
- Main Difference Lobe Width: $BW^{\Delta} = 0.38u$

Simulation Parameters:

- Sample Points: 2001
- Max Function Evaluations: 6000
- Max Iterations Number: 1000
- Function Tolerance: 1.0×10^{-8}
- Constraint Tolerance: 1×10^{-8}

Algorithm Behaviour:

- Simulation Time Pattern: 46 sec.

In the following figures are reported, for each iteration, the max values evaluated by the objective function and by the constraint function for the sum and difference pattern synthesis.

Sum Beam:

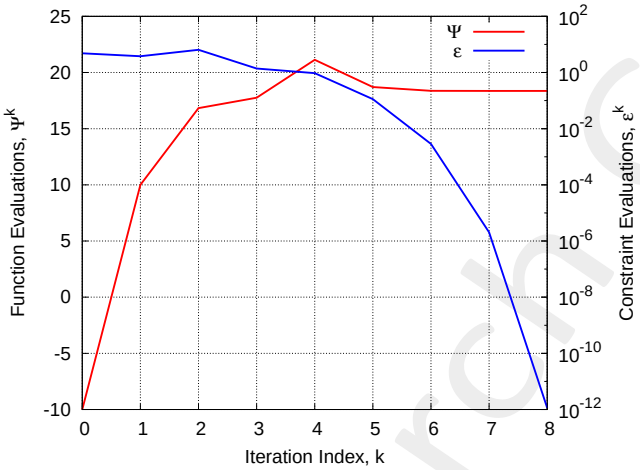

Figure 1. Sum Beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
18.36	1.1×10^{-12}

Table 1. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Difference Beam:

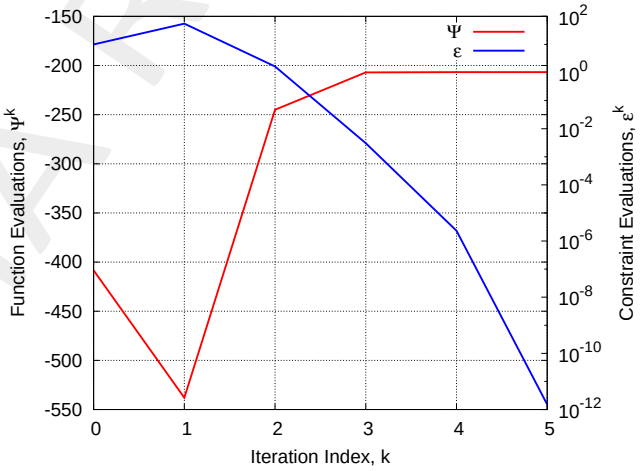

Figure 2. Difference beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
-206.7	1.5×10^{-12}

Table 2. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Excitations:

- Number of Common Elements: $P = 4$

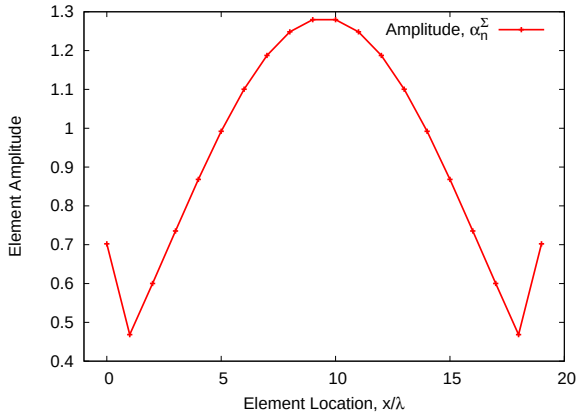

Figure 3. Sum beam's excitations amplitudes

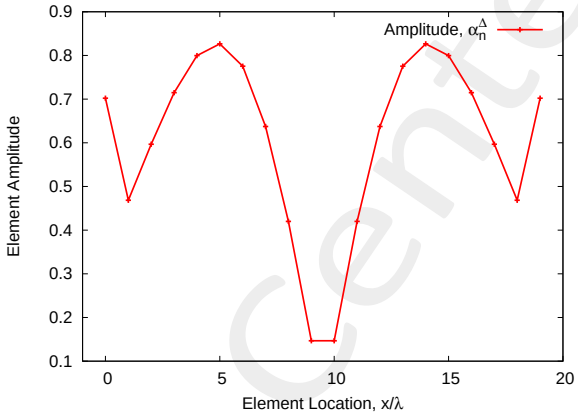

Figure 4. Difference beam's excitations amplitudes

Normalized Excitations:

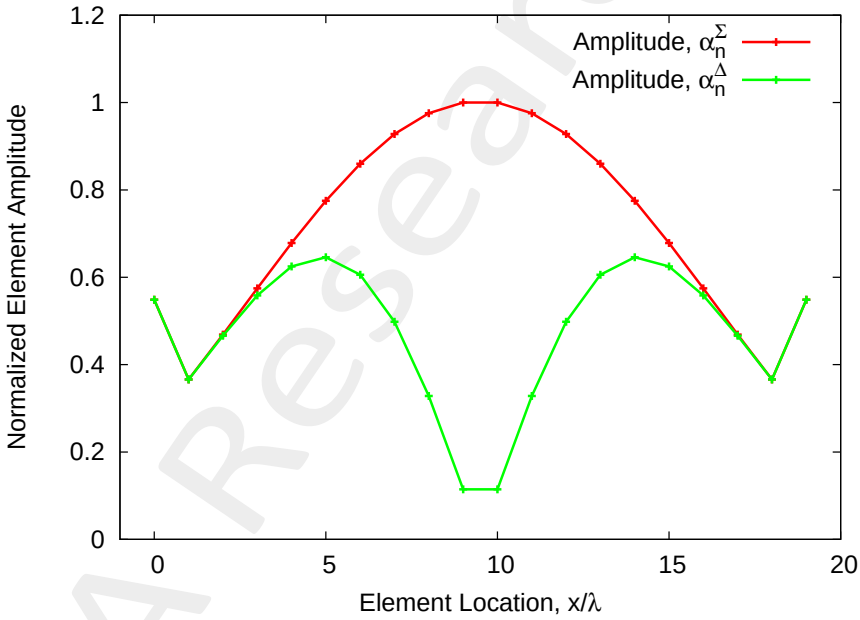

Figure 5. Sum and Difference beam's normalized excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
α_n^{Σ}	0.5487	0.366	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
α_n^{Δ}	0.5487	0.366	0.4663	0.5585	0.6249	0.6459	0.6059	0.4981	0.3281	0.1146

Table 3. Sum and Difference beam's nominal amplitudes values (symmetric excitations)

Sum Pattern:

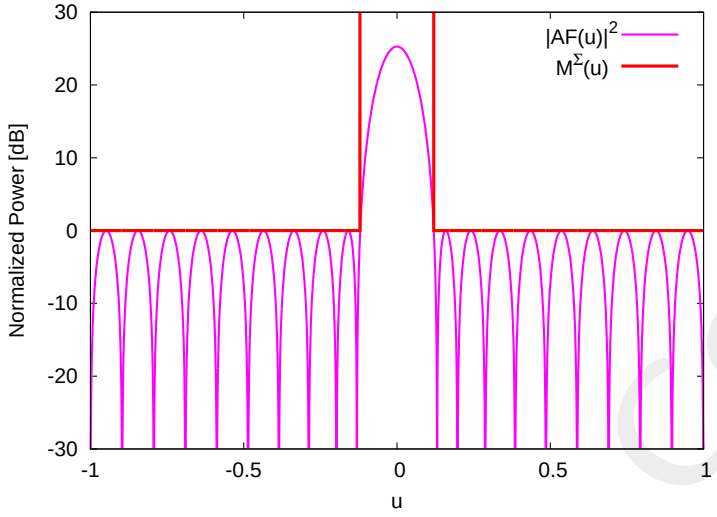

Figure 6. Synthesized Sum Pattern

SLL^{nom} [dB]	$D_{\max}^{nom}$ [dB]	BW^{nom} [u]	ψ_1 [u]
-25.28	12.65	0.104	0.131

Table 4. Sum beam's features values

Difference Pattern:

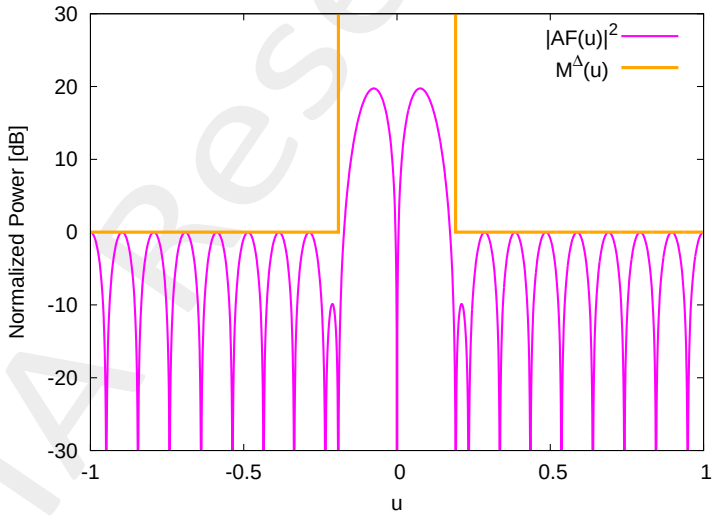

Figure 7. Synthesized Difference Pattern

SLL^{nom} [dB]	K^{nom} [1/rad]	BW^{nom} [u]	ψ_1 [u]
-19.74	1.2084	0.082	0.192

Table 5. Difference beam's features values

Tolerance Over Common Elements

- Total tolerance: $T = 3.6347$

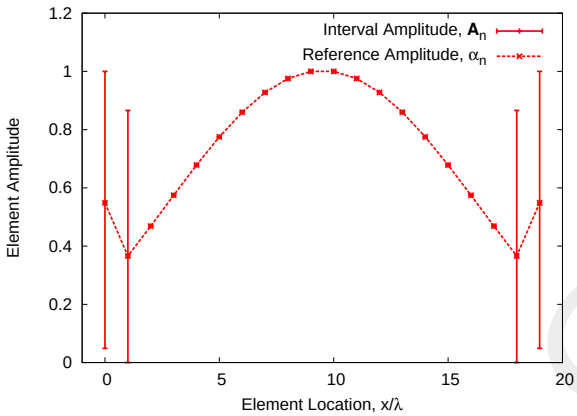

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$\inf\{\mathbf{A}_n\}$	0.0487	0.0	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$\sup\{\mathbf{A}_n\}$	1.0	0.866	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$w\{\mathbf{A}_n\}$	0.9513	0.866	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

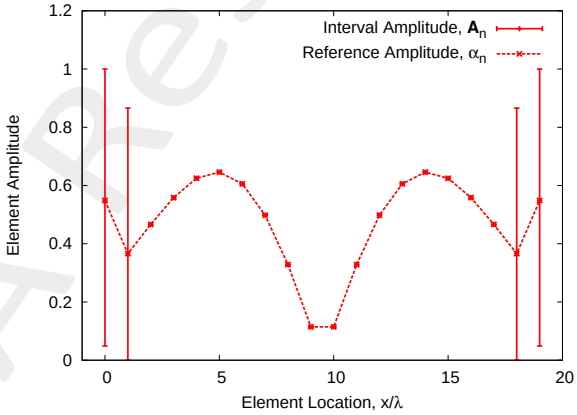

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4663	0.5585	0.6249	0.6459	0.6059	0.4981	0.3281	0.1146
$\inf\{\mathbf{A}_n\}$	0.0487	0.0	0.4663	0.5585	0.6249	0.6459	0.6059	0.4981	0.3281	0.1146
$\sup\{\mathbf{A}_n\}$	1.0	0.866	0.4663	0.5585	0.6249	0.6459	0.6059	0.4981	0.3281	0.1146
$w\{\mathbf{A}_n\}$	0.9513	0.866	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

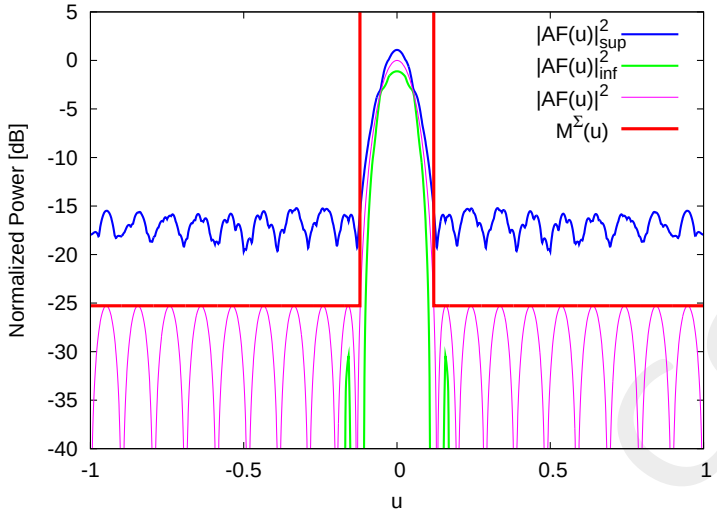

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.089	0.441
inf	-31.53	0.06	10.07	1.1		
sup	-14.1	0.138	14.02	-1.08		

Table 8. Sum Pattern Features

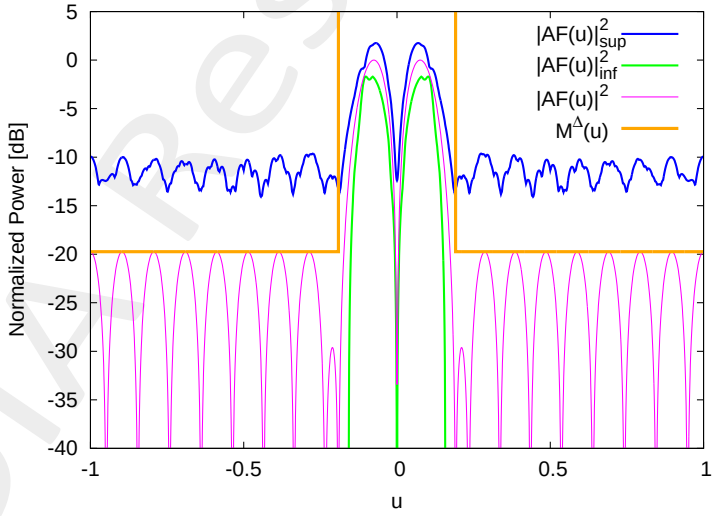

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-19.74	0.082	1.2084	0.0	0.2825	0.8826
inf	$-\infty$	0.074	0.6795	-1.7		
sup	-7.9	0.089	1.7695	1.75		

Table 9. Difference Pattern Features

Tolerance Over Not-Common Elements

- Total tolerance: $T = 3.6347$

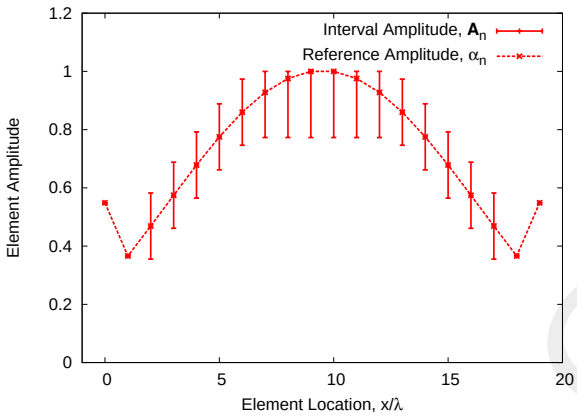

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$\inf\{\mathbf{A}_n\}$	0.5487	0.366	0.3553	0.461	0.5649	0.6616	0.7463	0.7728	0.7728	0.7728
$\sup\{\mathbf{A}_n\}$	0.5487	0.366	0.5824	0.6882	0.7921	0.8888	0.9735	1.0	1.0	1.0
$w\{\mathbf{A}_n\}$	0.0	0.0	0.2272	0.2272	0.2272	0.2272	0.2272	0.2272	0.2272	0.2272

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

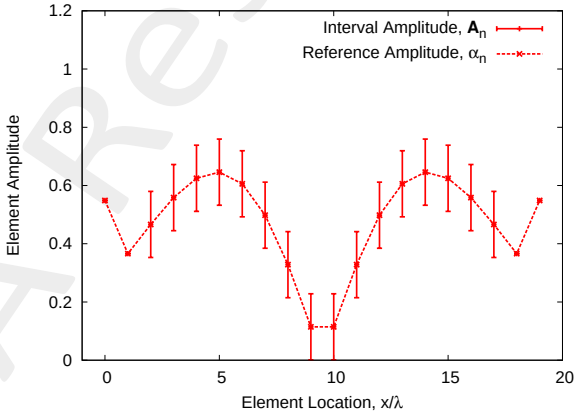

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4663	0.5585	0.6249	0.6459	0.6059	0.4981	0.3281	0.1146
$\inf\{\mathbf{A}_n\}$	0.5487	0.366	0.3527	0.4449	0.5113	0.5323	0.4923	0.3845	0.2145	0.001
$\sup\{\mathbf{A}_n\}$	0.5487	0.366	0.5798	0.6721	0.7385	0.7595	0.7195	0.6116	0.4417	0.2282
$w\{\mathbf{A}_n\}$	0.0	0.0	0.2272	0.2272	0.2272	0.2272	0.2272	0.2272	0.2272	0.2272

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

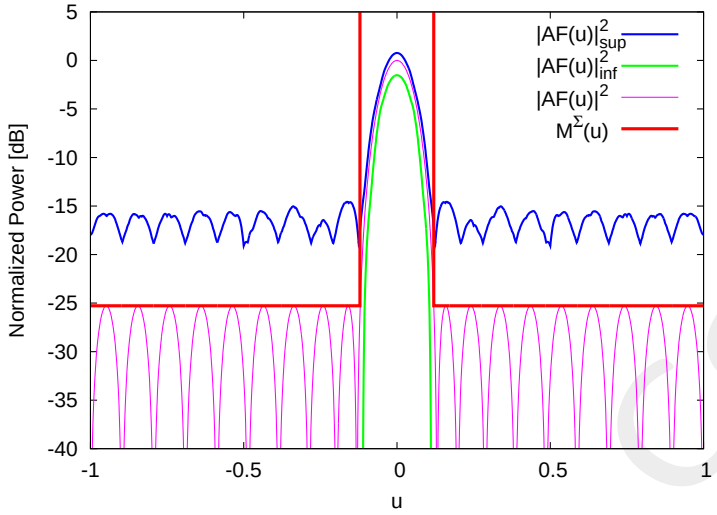

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.0995	0.4931
inf	$-\infty$	0.05	10.56	-1.52		
sup	-13.04	0.14	14.91	0.77		

Table 8. Sum Pattern Features

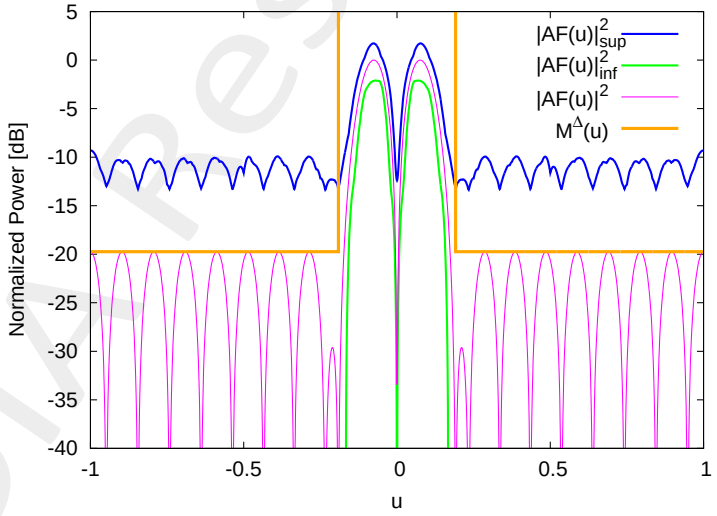

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-19.74	0.082	1.2084	0.0	0.2962	0.9254
inf	$-\infty$	0.078	0.8803	-2.11		
sup	-7.19	0.091	1.6397	1.72		

Table 9. Difference Pattern Features

1.3 Common Elements $P = 6$

Array Parameters:

- Number Elements: $N = 20$
- Services: Sum / Difference Beams
- Number of Common Elements: $P = 6$
- Element Spacing: $\lambda/2$

Constraints:

- Main Sum Lobe Width: $BW^{\Sigma} = 0.24u$
- Main Difference Lobe Width: $BW^{\Delta} = 0.38u$

Simulation Parameters:

- Sample Points: 2001
- Max Function Evaluations: 6000
- Max Iterations Number: 1000
- Function Tolerance: 1.0×10^{-8}
- Constraint Tolerance: 1×10^{-8}

Algorithm Behaviour:

- Simulation Time Pattern: 30 sec.

In the following figures are reported, for each iteration, the max values evaluated by the objective function and by the constraint function for the sum and difference pattern synthesis.

Sum Beam:

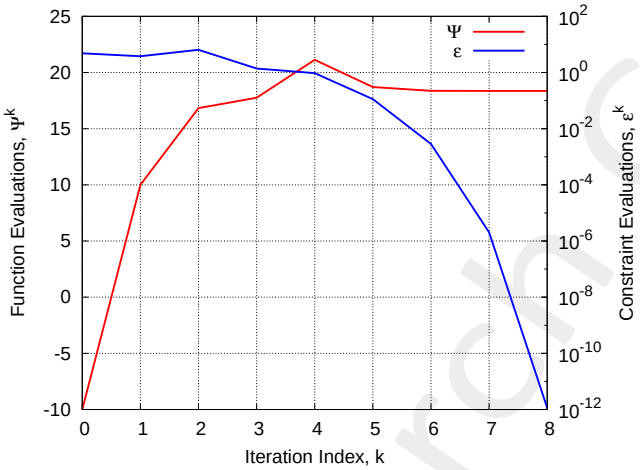

Figure 1. Sum Beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
-18.36	1.1×10^{-12}

Table 1. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Difference Beam:

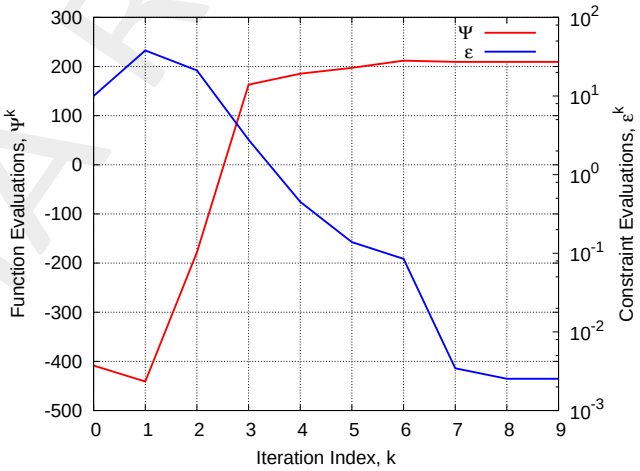

Figure 2. Difference beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
209.5	2.5×10^{-3}

Table 2. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Excitations:

- Number of Common Elements: $P = 6$

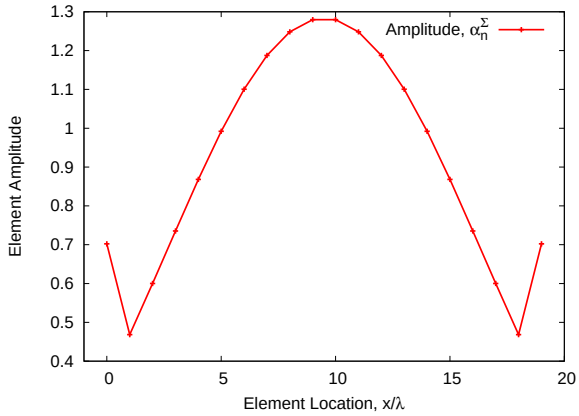

Figure 3. Sum beam's excitations amplitudes

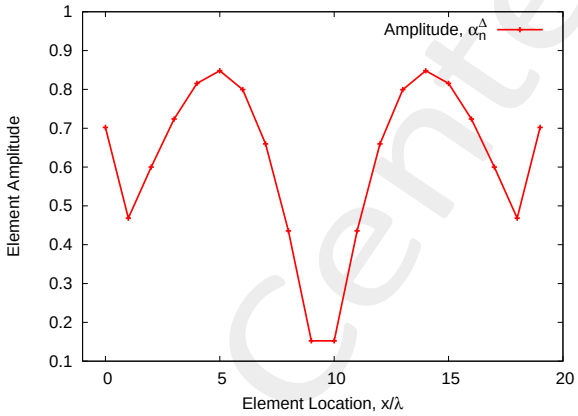

Figure 4. Difference beam's excitations amplitudes

Normalized Excitations:

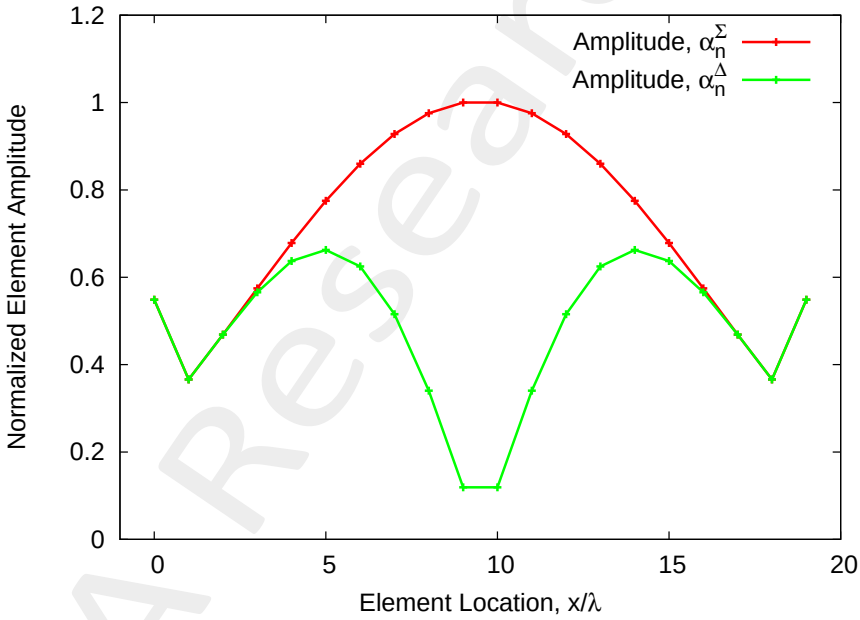

Figure 5. Sum and Difference beam's normalized excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
α_n^Σ	0.5487	0.366	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
α_n^Δ	0.5487	0.366	0.4689	0.5655	0.6372	0.6627	0.6247	0.5154	0.3404	0.119

Table 3. Sum and Difference beam's nominal amplitudes values (symmetric excitations)

Sum Pattern:

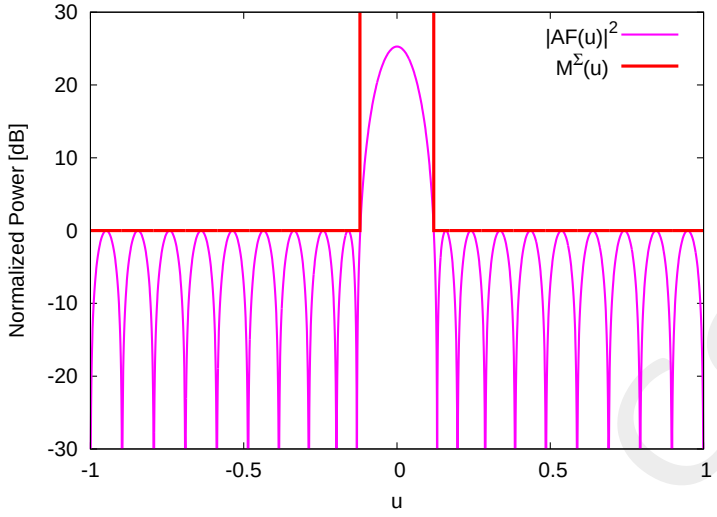

Figure 6. Synthesized Sum Pattern

SLL^{nom} [dB]	$D^{nom}_{\max}$ [dB]	BW^{nom} [u]	ψ_1 [u]
-25.28	12.65	0.104	0.131

Table 4. Sum beam's features values

Difference Pattern:

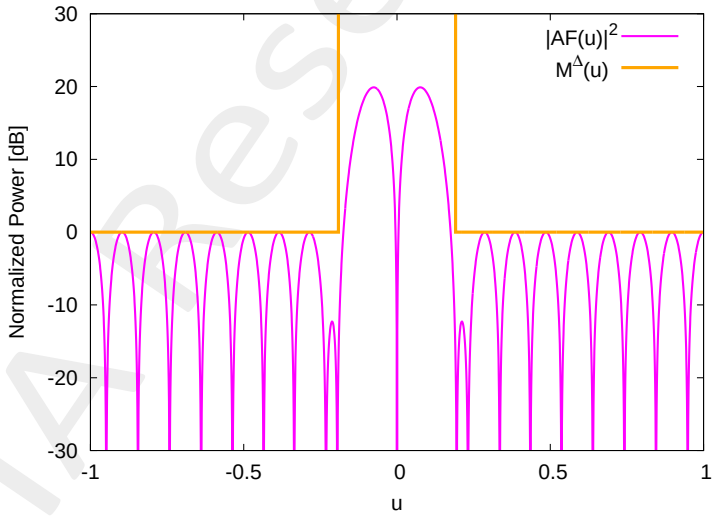

Figure 7. Synthesized Difference Pattern

SLL^{nom} [dB]	K^{nom} [1/rad]	BW^{nom} [u]	ψ_1 [u]
-19.8	1.2011	0.083	0.195

Table 5. Difference beam's features values

Tolerance Over Common Elements

- Total Tolerance: $T = 5.5724$

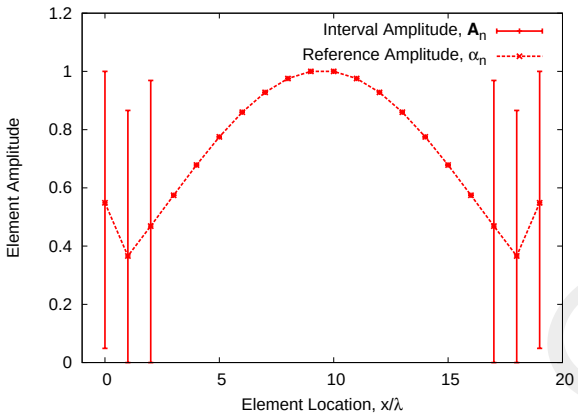

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$\inf\{\mathbf{A}_n\}$	0.0487	0.0	0.0	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$\sup\{\mathbf{A}_n\}$	1.0	0.866	0.9689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$w\{\mathbf{A}_n\}$	0.9513	0.866	0.9689	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

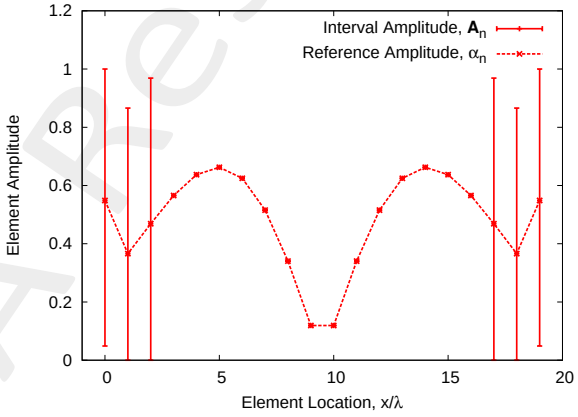

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5655	0.6372	0.6627	0.6247	0.5154	0.3404	0.119
$\inf\{\mathbf{A}_n\}$	0.0487	0.0	0.0	0.5655	0.6372	0.6627	0.6247	0.5154	0.3404	0.119
$\sup\{\mathbf{A}_n\}$	1.0	0.866	0.9687	0.5655	0.6372	0.6627	0.6247	0.5154	0.3404	0.119
$w\{\mathbf{A}_n\}$	0.9513	0.866	0.9689	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

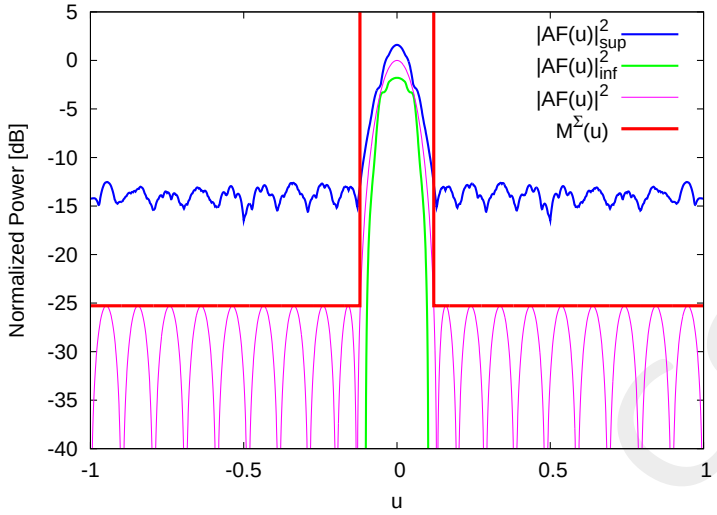

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.1564	0.7746
inf	$-\infty$	0.0	9.68	-1.79		
sup	-10.74	0.158	14.71	1.6		

Table 8. Sum Pattern Features

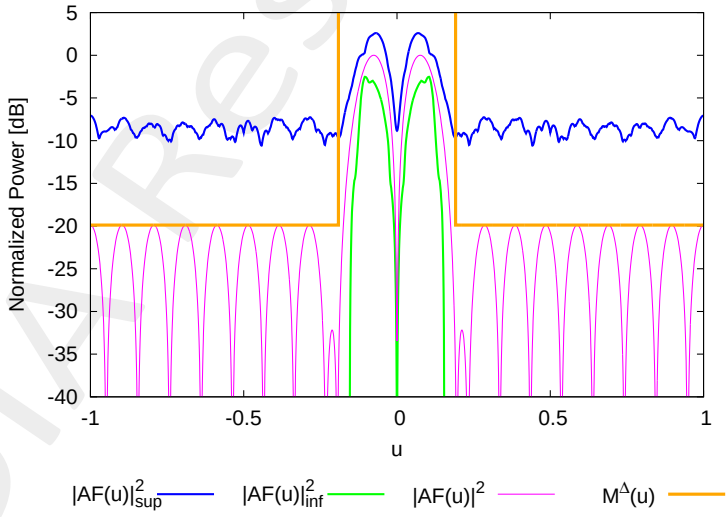

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-19.8	0.083	1.2011	0.0	0.4919	1.5388
inf	$-\infty$	0.0	0.5069	-2.52		
sup	-4.55	0.17	2.0416	2.6		

Table 9. Difference Pattern Features

Tolerance Over Not-Common Elements

- Total Tolerance: $T = 5.5724$

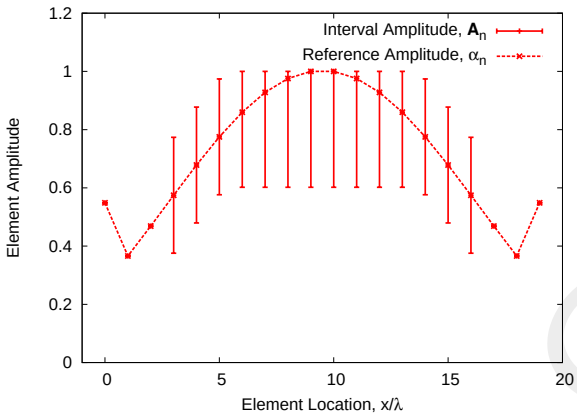

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$\inf\{\mathbf{A}_n\}$	0.5487	0.366	0.4689	0.3756	0.4794	0.5762	0.602	0.602	0.602	0.602
$\sup\{\mathbf{A}_n\}$	0.5487	0.366	0.4689	0.7736	0.8775	0.9742	1.0	1.0	1.0	1.0
$w\{\mathbf{A}_n\}$	0.0	0.0	0.0	0.398	0.398	0.398	0.398	0.398	0.398	0.398

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

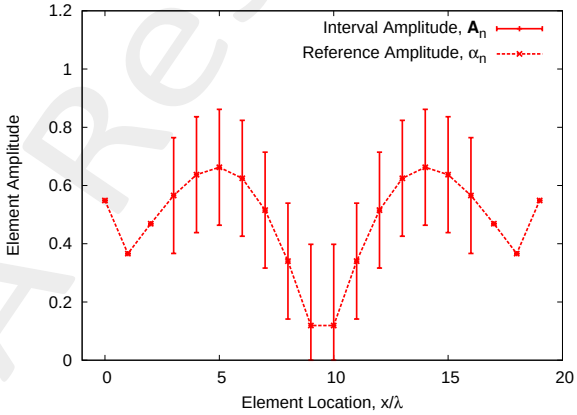

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5655	0.6372	0.6627	0.6247	0.5154	0.3404	0.119
$\inf\{\mathbf{A}_n\}$	0.5487	0.366	0.4689	0.3665	0.4382	0.4637	0.4257	0.3164	0.1414	0.0
$\sup\{\mathbf{A}_n\}$	0.5487	0.366	0.4689	0.7645	0.8362	0.8617	0.8238	0.7144	0.5394	0.398
$w\{\mathbf{A}_n\}$	0.0	0.0	0.0	0.398	0.398	0.398	0.398	0.398	0.398	0.398

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

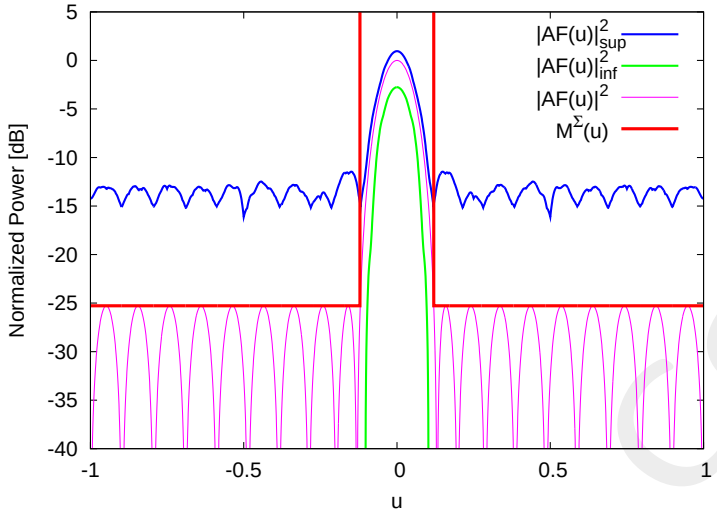

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.1671	0.8275
inf	$-\infty$	0.0	9.25	-2.76		
sup	-8.7	0.16	16.46	0.95		

Table 8. Sum Pattern Features

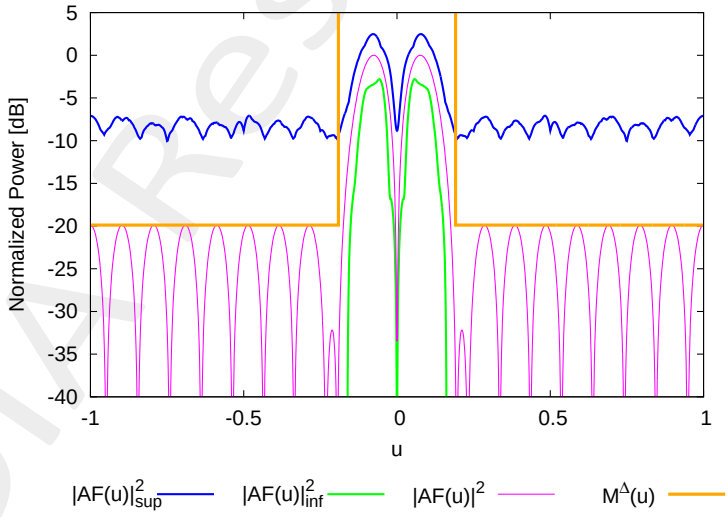

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-19.8	0.083	1.2011	0.0	0.506	1.5831
inf	$-\infty$	0.0	0.7597	-2.8		
sup	-4.28	0.161	1.8833	2.5		

Table 9. Difference Pattern Features

1.4 Common Elements $P = 8$

Array Parameters:

- Number Elements: $N = 20$
- Services: Sum / Difference Beams
- Number of Common Elements: $P = 8$
- Element Spacing: $\lambda/2$

Constraints:

- Main Sum Lobe Width: $BW^{\Sigma} = 0.24u$
- Main Difference Lobe Width: $BW^{\Delta} = 0.38u$

Simulation Parameters:

- Sample Points: 2001
- Max Function Evaluations: 6000
- Max Iterations Number: 1000
- Function Tolerance: 1.0×10^{-8}
- Constraint Tolerance: 1×10^{-8}

Algorithm Behaviour:

- Simulation Time Pattern: 30 sec.

In the following figures are reported, for each iteration, the max values evaluated by the objective function and by the constraint function for the sum and difference pattern synthesis.

Sum Beam:

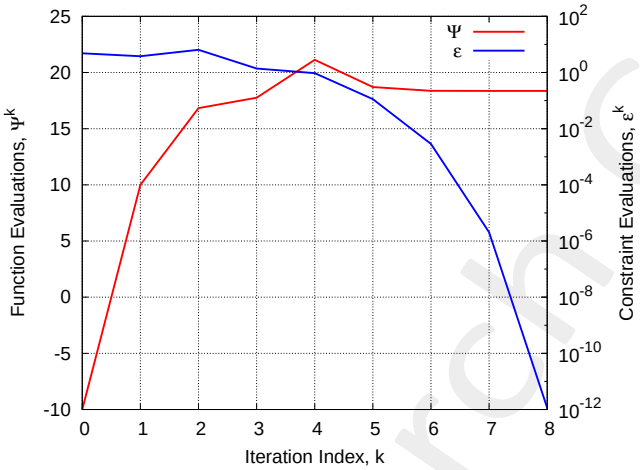

Figure 1. Sum Beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
-18.36	1.1×10^{-12}

Table 1. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Difference Beam:

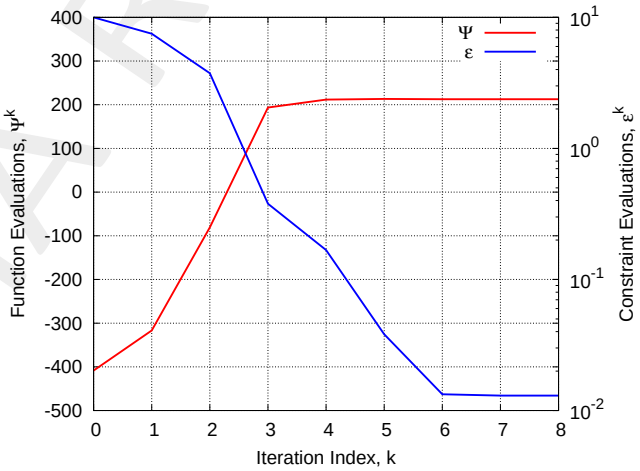

Figure 2. Difference beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
212.7	1.3×10^{-2}

Table 2. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Excitations:

- Number of Common Elements: $P = 8$

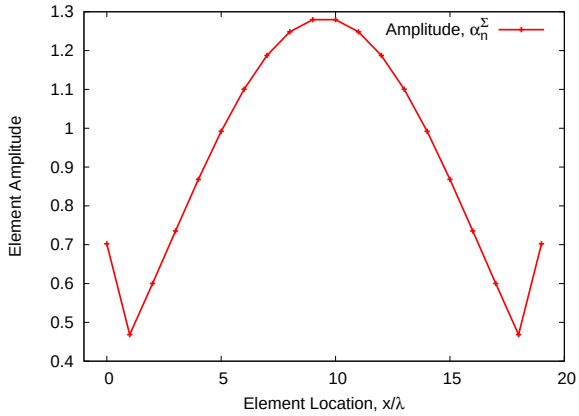

Figure 3. Sum beam's excitations amplitudes

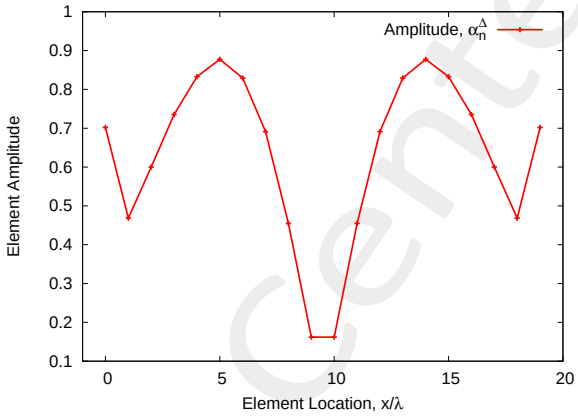

Figure 4. Difference beam's excitations amplitudes

Normalized Excitations:

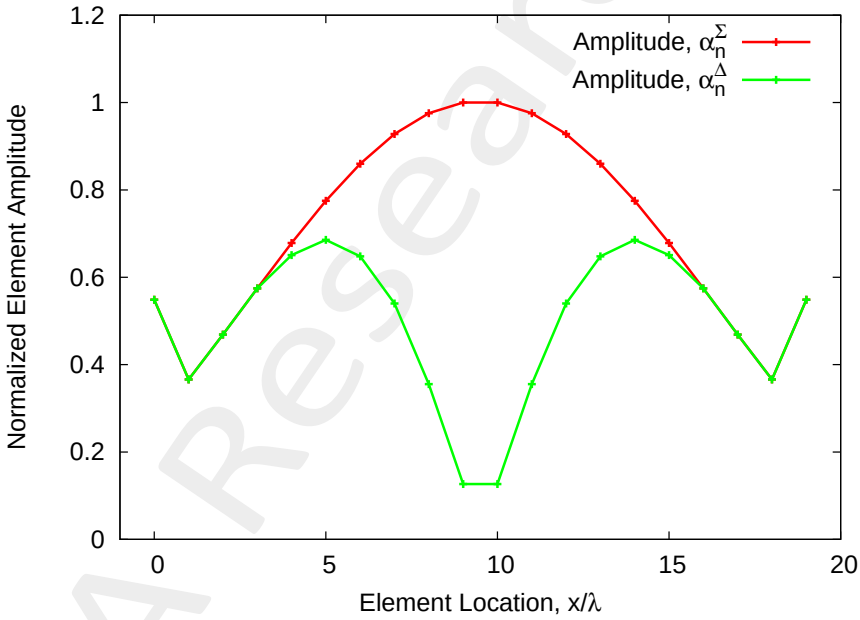

Figure 5. Sum and Difference beam's normalized excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
α_n^{Σ}	0.5487	0.366	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
α_n^{Δ}	0.5487	0.366	0.4689	0.5746	0.6507	0.6858	0.648	0.5401	0.3554	0.1265

Table 3. Sum and Difference beam's nominal amplitudes values (symmetric excitations)

Sum Pattern:

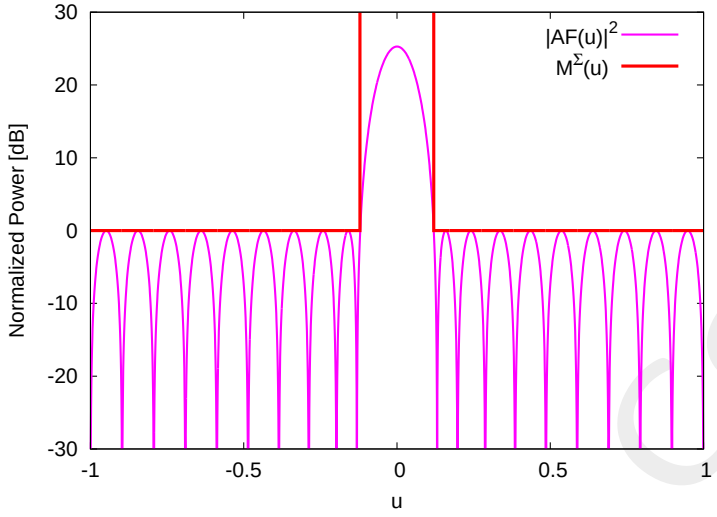

Figure 6. Synthesized Sum Pattern

SLL^{nom} [dB]	$D_{\max}^{nom}$ [dB]	BW^{nom} [u]	ψ_1 [u]
-25.28	12.65	0.104	0.131

Table 4. Sum beam's features values

Difference Pattern:

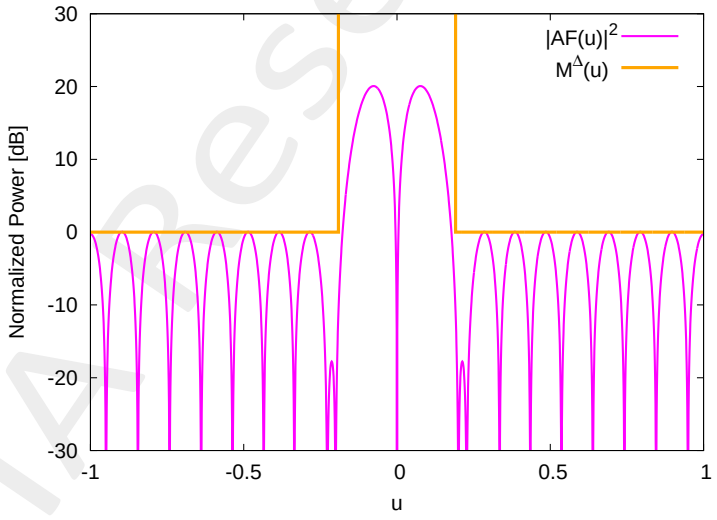

Figure 7. Synthesized Difference Pattern

SLL^{nom} [dB]	K^{nom} [1/rad]	BW^{nom} [u]	ψ_1 [u]
-20.01	1.1912	0.83	0.2

Table 5. Difference beam's features values

Tolerance Over Common Elements

- Total tolerance: $T = 7.4232$

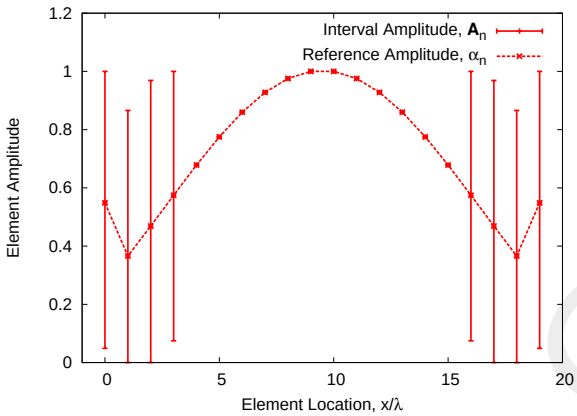

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$\inf\{\mathbf{A}_n\}$	0.0487	0.0	0.0	0.0746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$\sup\{\mathbf{A}_n\}$	1.0	0.866	0.9688	1.0	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$w\{\mathbf{A}_n\}$	0.9513	0.866	0.9689	0.9254	0.0	0.0	0.0	0.0	0.0	0.0

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

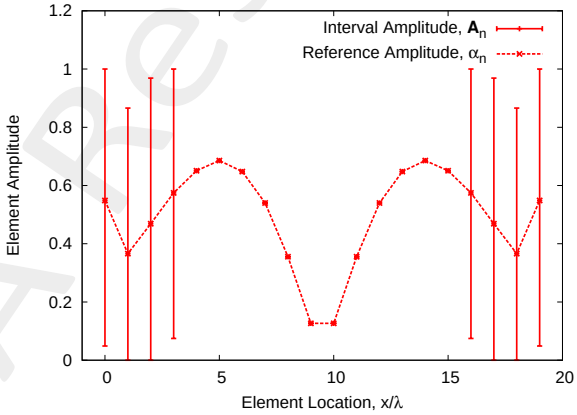

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5746	0.6507	0.6858	0.648	0.5401	0.3554	0.1265
$\inf\{\mathbf{A}_n\}$	0.0487	0.0	0.0	0.0746	0.6507	0.6858	0.648	0.5401	0.3554	0.1265
$\sup\{\mathbf{A}_n\}$	1.0	0.866	0.9689	1.0	0.6507	0.6858	0.648	0.5401	0.3554	0.1265
$w\{\mathbf{A}_n\}$	0.9513	0.866	0.9689	0.9254	0.0	0.0	0.0	0.0	0.0	0.0

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

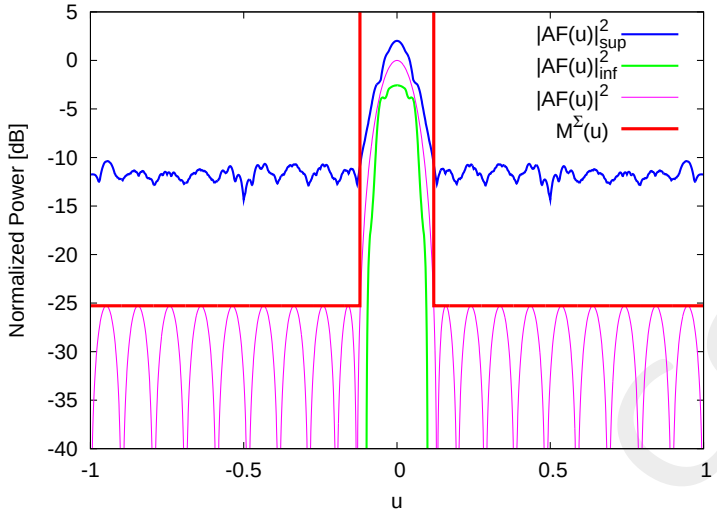

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.2342	1.1603
inf	$-\infty$	0.0	8.64	-2.57		
sup	-7.8	0.178	15.4	2.02		

Table 8. Sum Pattern Features

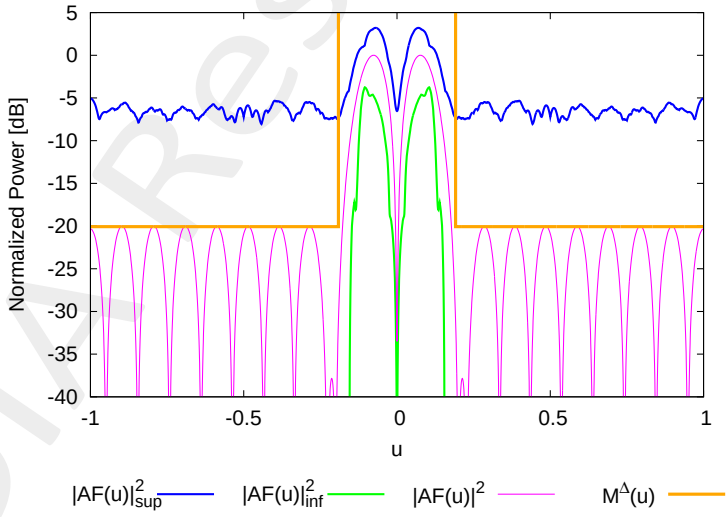

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-20.01	0.083	1.1912	0.0	0.72	2.251
inf	$-\infty$	–	0.3776	-3.75		
sup	-1.3	4.0	2.3317	3.21		

Table 9. Difference Pattern Features

Tolerance Over Not-Common Elements

- Total tolerance: $T = 7.4232$

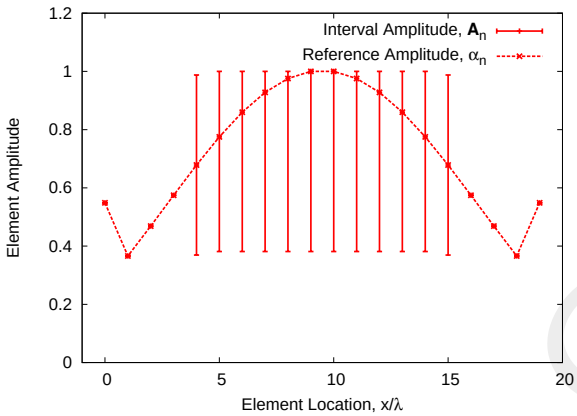

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5746	0.6785	0.7752	0.8599	0.9279	0.9755	1.0
$\inf\{\mathbf{A}_n\}$	0.5487	0.366	0.4689	0.5746	0.3692	0.3814	0.3814	0.3814	0.3814	0.3814
$\sup\{\mathbf{A}_n\}$	0.5487	0.366	0.4689	0.5746	0.9878	1.0	1.0	1.0	1.0	1.0
$w\{\mathbf{A}_n\}$	0.0	0.0	0.0	0.0	0.6186	0.6186	0.6186	0.6186	0.6186	0.6186

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

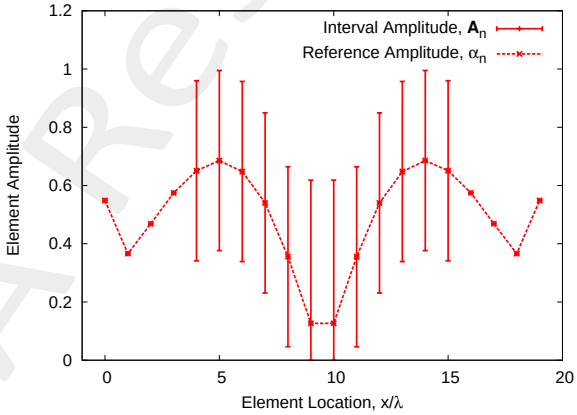

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.5487	0.366	0.4689	0.5746	0.6507	0.6858	0.648	0.5401	0.3554	0.1265
$\inf\{\mathbf{A}_n\}$	0.5487	0.366	0.4689	0.5746	0.3414	0.3765	0.3387	0.2308	0.0461	0.0
$\sup\{\mathbf{A}_n\}$	0.5487	0.366	0.4689	0.5746	0.96	0.9951	0.9573	0.8494	0.6647	0.6186
$w\{\mathbf{A}_n\}$	0.0	0.0	0.0	0.0	0.6186	0.6186	0.6186	0.6186	0.6186	0.6186

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

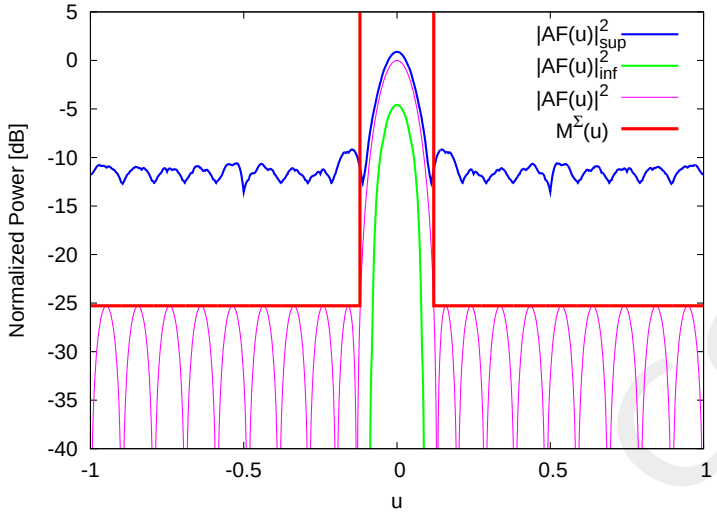

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.2373	1.1754
inf	$-\infty$	0.0	7.7	-4.58		
sup	-4.6	0.178	18.17	0.87		

Table 8. Sum Pattern Features

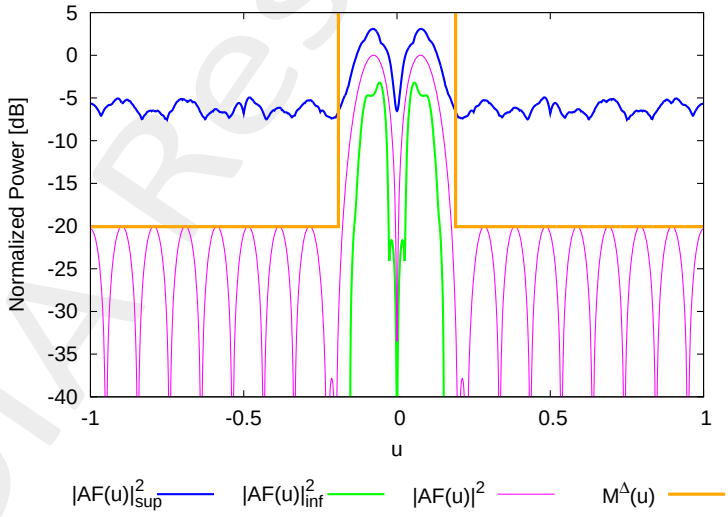

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-20.01	0.0	1.1912	0.0	0.7393	2.3112
inf	$-\infty$	0.0	0.6745	-3.19		
sup	-1.7	0.428	2.0105	3.09		

Table 9. Difference Pattern Features

More information on the topics of this document can be found in the following list of references.

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