Interval Arithmetic for Assessing Tolerance in Reconfigurable Monopulse Linear Antenna Arrays

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Contents

1 Introduction

Reconfigurable antenna arrays are able to dynamically change the radiation behaviour by just modifying the excitations' weights, affording different services. This can be done by using an independent feeding network for each service, and switch among them to generate the corresponding beam pattern; or by sharing some excitations and switch just the remaining weights. In order to reduce the cost of the antenna's implementation, the main issue is to maximize the number of common control points, on the other end these elements are more critical in terms of manufacturing tolerances element's malfunctions and failures. Interval Analysis is here applied in order to investigate the critically of the common control points with respect to the others.

2 Variable Definition

2.1 Interval Analysis

- **Real Interval Numbers (Intervals)**

  A closed real interval $\mathbf{X} = [a,\, b]$, namely interval number $\mathbf{X}$, consists of the set of real numbers x such that $\{x :\, a \leq x \leq b\}$, i.e. it is the set of real numbers between and including the endpoints a and b. Consequently a real number x is equivalent to an interval $[x,\, x]$. Such an interval is said to be *degenerate* interval. A superscript L (or R) denotes the left (or right) endpoint of an interval. Thus if $\mathbf{X} = [a,\, b]$ then $\inf\{\mathbf{X}\} = a$ and $\sup\{\mathbf{X}\} = b$. Consequently, the right endpoint of the interval $\mathbf{X}$ is usually denoted as $\sup\{\mathbf{X}\}$ and the left endpoint of the interval is usually denoted as $\inf\{\mathbf{X}\}$.

  An interval $\mathbf{X} = [a, b]$ is said to be *positive* (or *nonnegative*) if $a \geq 0$, *strictly positive* if $a > 0$, *negative* (or *nonpositive*) if $b \leq 0$, and *strictly negative* if $b < 0$. Two intervals $[a, b]$ and $[c, d]$ are *equal* if and only if $a = c$ and $b = d$. Intervals are partially ordered. $[a, b] < [c, d] \Leftrightarrow b < c$.

- **Interval Arithmetic**

  Let us denote $+$, $-$, $*$ and $/$ the operation of addition, subtraction, multiplication and division, respectively. Let be *op* any one of these operations for the arithmetic of real numbers x and y, then the corresponding operation for the arithmetic of interval numbers $\mathbf{X}$ and $\mathbf{Y}$ is:

$$\mathbf{X}\, op\, \mathbf{Y} = \{x\, op\, y \,:\, x \in \mathbf{X},\, y \in \mathbf{Y}\}\,. \tag{1}$$

  From definition (1) follows that the interval $\mathbf{X}\, op\, \mathbf{Y}$ contains every possible number which can be formed as $x\, op\, y$ for each $x \in \mathbf{X}$ and $y \in \mathbf{Y}$. Moreover, following such a definition it is possible to produce the rules for generating the endpoints of $\mathbf{X}\, op\, \mathbf{Y}$ from the two interval operands $\mathbf{X}$ and $\mathbf{Y}$.

- **Interval Sum**

  Let be $\mathbf{X} = [\inf\{\mathbf{X}\}\,,\, \sup\{\mathbf{X}\}]$ and $\mathbf{Y} = [\inf\{\mathbf{Y}\}\,,\, \sup\{\mathbf{Y}\}]$ two real intervals. The sum operation is defined as:

$$\mathbf{X} + \mathbf{Y} = [\inf\{\mathbf{X}\} + \inf\{\mathbf{Y}\}\,,\, \sup\{\mathbf{X}\} + \sup\{\mathbf{Y}\}] \tag{2}$$

- **Interval Subtraction**

  Let be $\mathbf{X} = [\inf\{\mathbf{X}\}\,,\, \sup\{\mathbf{X}\}]$ and $\mathbf{Y} = [\inf\{\mathbf{Y}\}\,,\, \sup\{\mathbf{Y}\}]$ two real intervals. The subtraction operation is defined as:

$$\mathbf{X} - \mathbf{Y} = [\inf\{\mathbf{X}\} - \inf\{\mathbf{Y}\}\,,\, \sup\{\mathbf{X}\} - \sup\{\mathbf{Y}\}] \tag{3}$$

- **Interval Multiplication**

Let be $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ and $\mathbf{Y} = [\inf\{\mathbf{Y}\}, \sup\{\mathbf{Y}\}]$ two real intervals. The multiplication operation is defined as:

$$\begin{aligned} \mathbf{X} * \mathbf{Y} \quad = \quad & [\min(\inf\{\mathbf{X}\}\inf\{\mathbf{Y}\}, \inf\{\mathbf{X}\}\inf\{\mathbf{Y}\}, \mathbf{X}_{\sup}\inf\{\mathbf{Y}\}, \sup\{\mathbf{X}\}\sup\{\mathbf{Y}\}), \\ & \max(\inf\{\mathbf{X}\}\inf\{\mathbf{Y}\}, \inf\{\mathbf{X}\}\sup\{\mathbf{Y}\}, \mathbf{X}_{\sup}\inf\{\mathbf{Y}\}, \sup\{\mathbf{X}\}\sup\{\mathbf{Y}\})] \end{aligned} \tag{4}$$

- **Interval Inverse**

  Let be$\mathbf{Y} = [\inf\{\mathbf{Y}\}, \sup\{\mathbf{Y}\}]$ a real interval. The inverse interval $1/\mathbf{Y}$ is defined as:

$$\frac{1}{\mathbf{Y}} = \left[\frac{1}{\sup\{\mathbf{Y}\}}, \frac{1}{\inf\{\mathbf{Y}\}}\right] \; 0 \notin \mathbf{Y} \tag{5}$$

- **Interval Division**

  Let be $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ and $\mathbf{Y} = [\inf\{\mathbf{Y}\}, \sup\{\mathbf{Y}\}]$ two real intervals. The interval division is defined by means of interval inverse and interval multiplication as:

$$\frac{\mathbf{X}}{\mathbf{Y}} = \mathbf{X} * \left(\frac{1}{\mathbf{Y}}\right) \; 0 \notin \mathbf{Y} \tag{6}$$

- **Power of a Real Interval**

  Let be $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ a real interval. The power of $\mathbf{X}$, $\mathbf{X}^n$ is a real interval computed as:

$$\mathbf{X}^n = \begin{cases} [1, 1] & if\, n = 0 \\ [\inf\{\mathbf{X}^n\}, \sup\{\mathbf{X}^n\}] & if\ \inf\{\mathbf{X}\} \geq 0\, or\, if\ \inf\{\mathbf{X}\} \leq 0 \leq \sup\{\mathbf{X}\}\ and\, n\, is\, odd \\ [\sup\{\mathbf{X}^n\}, \inf\{\mathbf{X}^n\}] & if\ \sup\{\mathbf{X}\} \leq 0 \\ [0, \max(\inf\{\mathbf{X}^n\}, \sup\{\mathbf{X}^n\})] & if\ \inf\{\mathbf{X}\} \leq 0 \leq \sup\{\mathbf{X}\}\ and\, n\, is\, even \end{cases} \tag{7}$$

- **Square of a Real Interval**

  Let be $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ a real interval. Accordingly to equation (7) the square of the interval $\mathbf{X}$, $\mathbf{X}^2$ is an interval computed as:

$$\mathbf{X}^2 = \begin{cases} [\inf\{\mathbf{X}^2\}, \sup\{\mathbf{X}^2\}] & if\ \inf\{\mathbf{X}\} \geq 0 \\ [\sup\{\mathbf{X}^2\}, \inf\{\mathbf{X}^2\}] & if\ \sup\{\mathbf{X}\} \leq 0 \\ [0, \max(\inf\{\mathbf{X}^n\}, \sup\{\mathbf{X}^n\})] & if\ \inf\{\mathbf{X}\} \leq 0 \leq \sup\{\mathbf{X}\} \end{cases} \tag{8}$$

- **Width of a Real Interval**

  Let be $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ a real interval. The width of the interval $\mathbf{X}$, $w(\mathbf{X})$ is a real number defined as:

$$w(\mathbf{X}) = \sup\{\mathbf{X}\} - \inf\{\mathbf{X}\} \tag{9}$$

- **Center of a Real Interval - Middle point**

Let be $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ a real interval. The middle point of the interval $\mathbf{X}$, $m(\mathbf{X})$ is defined as:

$$m(\mathbf{X}) = \frac{\sup\{\mathbf{X}\} + \inf\{\mathbf{X}\}}{2} \tag{10}$$

- **Equivalent representation of Intervals**

  Let be $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ a real interval, it can be represent equivalently in the form:

$$\mathbf{X} = \left[m(\mathbf{X}) - \frac{w(\mathbf{X})}{2};\, m(\mathbf{X}) + \frac{w(\mathbf{X})}{2}\right] \tag{11}$$

- **Complex Interval Numbers (Intervals)**

  A complex intervals $\mathbf{C}$ is an ordered pair of intervals $\mathbf{C} = [\mathbf{X}, \mathbf{Y}]$ with $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ and $\mathbf{X} = [\inf\{\mathbf{Y}\}, \sup\{\mathbf{Y}\}]$real intervals. It consists in the set of complex numbers $x + iy$ such that:

$$\mathbf{X} = \{x + iy \inf\{\mathbf{X}\} \leq x \leq \sup\{\mathbf{X}\} \inf\{\mathbf{Y}\} \leq y \leq \sup\{\mathbf{Y}\}\} \tag{12}$$

- **Negative of a Complex Interval**

  Let be $\mathbf{C} = [\mathbf{X}, \mathbf{Y}]$ with $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ and $\mathbf{Y} = [\inf\{\mathbf{Y}\}, \sup\{\mathbf{Y}\}]$real intervals. The negative of $\mathbf{C}$, $-\mathbf{C}$ is defined as:

$$-\mathbf{C} = [-\mathbf{X}, -\mathbf{Y}] \tag{13}$$

  where in equation (13), $-\mathbf{X}$ is the real interval $-\mathbf{X} = [-\sup\{\mathbf{X}\}, -\inf\{\mathbf{X}\}]$ and $-\mathbf{Y}$ is the real interval $-\mathbf{Y} = [-\sup\{\mathbf{Y}\}, -\inf\{\mathbf{Y}\}]$.

- **Complex Conjugate Complex Interval**

  Let be $\mathbf{C} = [\mathbf{X}, \mathbf{Y}]$ with $\mathbf{X} = [\inf\{\mathbf{X}\}, \sup\{\mathbf{X}\}]$ and $\mathbf{X} = [\inf\{\mathbf{Y}\}, \sup\{\mathbf{Y}\}]$real intervals. The complex conjugate of the complex interval $\mathbf{C}$, $\mathbf{C}*$ is defined as:

$$\mathbf{C}^* = [\mathbf{X}, -\mathbf{Y}] \tag{14}$$

- **Sum of Complex Intervals**

  Let be $\mathbf{C} = [\mathbf{X}, \mathbf{Y}]$ and $\mathbf{C}' = [\mathbf{X}', \mathbf{Y}']$ two complex intervals. The sum of the complex intervals is a complex interval defined as:

$$\mathbf{C} + \mathbf{C}' = [\mathbf{X} + \mathbf{X}', \mathbf{Y} + \mathbf{Y}'] \tag{15}$$

- **Product of Complex Intervals**

  Let be $\mathbf{C} = [\mathbf{X}, \mathbf{Y}]$ and $\mathbf{C}' = [\mathbf{X}', \mathbf{Y}']$ two complex intervals. The product of the complex intervals is a complex interval defined as:

$$\mathbf{C} \cdot \mathbf{C}' = [\mathbf{XX}' - \mathbf{YY}', \, \mathbf{XY}' + \mathbf{YX}'] \tag{16}$$

- **Sum of a complex interval and its complex conjugate**

  Let be $\mathbf{C} = [\mathbf{X}, \, \mathbf{Y}]$ a complex interval and $\mathbf{C}* = [\mathbf{X}, \, \mathbf{Y}]$ its complex conjugate. The sum $\mathbf{C}$ and $\mathbf{C}*$ is the real interval defined as:

$$\mathbf{X} + \mathbf{X}^{*} = [2\mathbf{X}] \tag{17}$$

- **Product of a complex interval and its complex conjugate**

  Let be $\mathbf{C} = [\mathbf{X}, \, \mathbf{Y}]$ a complex interval and $\mathbf{C}* = [\mathbf{X}, \, \mathbf{Y}]$ its complex conjugate. The product of $\mathbf{C}$ and its complex conjugate $\mathbf{C}*$ is the real interval:

$$\mathbf{X} \cdot \mathbf{X}* = \left[\mathbf{X^2} + \mathbf{Y^2}\right] \tag{18}$$

2.2 Array Synthesis and Analysis

- **Array Factor:**

  The array factor of an uniform linear array is defined as:

$$AF(\theta) = \sum_{n=1}^{N} a_n e^{j(n-1)k \cdot d \sin(\theta)} \, \theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \tag{19}$$

  where $a_n;\ n = 1, \ldots, N$ are the excitations coefficients of the N elements of the array, d is the element spacing, θ is the angle measured from the array axis.

- **Array Excitations:**

$$a_n = \alpha_n \cdot e^{j\beta_n} \tag{20}$$

  where $\alpha_n;\ n = 1, \ldots, N$ are the amplitudes and $\beta_n;\ n = 1, \ldots, N$ are the phases of the $a_n;\ n = 1, \ldots, N$ excitation coefficients.

- **Variable u:**

$$u = \sin\theta \, \theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \tag{21}$$

- **Normalized Array Factor:**

$$AF(\theta)_n = \frac{AF(\theta)}{\max_\theta \{|AF(\theta)|\}} \, \theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \tag{22}$$

  where in (22) $\max_\theta \{|AF(\theta)|\}$ is the maximal value of $|AF(\theta)| \ \theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$.

- **Position of the maximum of AF:**

$$u_{\max} = u \in [-1; 1] \ AF(u_{\max}) = \max\{AF(u)\} \ u \in [-1; 1] \tag{23}$$

- **Power Pattern:**

  The power pattern radiated by an antenna array is computed as follows:

$$P(\theta) = AF(\theta)\, AF(\theta)^* = |AF(\theta)|^2 = Re^2\{AF(\theta)\} + Im^2\{AF(\theta)\} \tag{24}$$

  where in equation (24) $AF(\theta)^*$ is the complex-conjugate of the $AF(\theta)$.

- $3\,[dB]$ **Beam Width** $HPBW$**:**

$$HPBW = \theta_A - \theta_B \tag{25}$$

where $\theta_A$and θ_B are the angular positions where $|AF(\theta)|_n^2 = -3\,[dB]$

- **Normalized Power Level P_{RL}:**

$$P_{RL}(\theta) = 20 \cdot \log_{10} |AF(\theta)|_n \tag{26}$$

- **First Null Position ψ_1:**

  Position of the first null of $|AF(\theta)|_n$

- **Maximal SLL $\max\{SLL\}$:**

  It is the maximal level of the grating lobes of $|AF(\theta)|_n$.

- **Maximal $Directivity$ - D_{max}:**

  The directivity is defined as:

$$D(\theta, \phi) = \frac{U(\theta, \phi)}{\frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} U(\theta, \phi) \sin(\theta)\, d\theta d\phi} \tag{27}$$

where $U(\theta, \phi)$ is the radiation intensity:

$$U(\theta, \phi) = \frac{r^2}{2\eta} |\underline{E}_{tot}(\underline{r}, \theta, \phi)|^2 = U_0(\theta, \phi)\, |AF(\theta, \phi)|^2 \tag{28}$$

$U_0(\theta, \phi)$ is radiation intensity for the single element. In our case, considering ideal isotropic sources, $U_0(\theta, \phi) = 1$. Usually "peak directivity" is more used, that is the directivity calculated at its maximum: $D_{max}(\theta_0, \phi_0)$. For our array, the direction of the maximum is at $\theta = 0$:

$$U(\theta_0, \phi_0) = |AF(\theta_0, \phi_0)|^2 = \left| \sum_{n=1}^{N} a_n e^{j(n-1)k \cdot d \cdot \sin(\theta_0)} \right|^2 = \left| \sum_{n=0}^{N-1} a_n \right|^2 \tag{29}$$

Now, computing the denominator of (27), we obtain:

$$\frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} U(\theta, \phi) sin(\theta) d\theta d\phi = \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m^* \int_{-1}^{1} e^{jkd(n-m)u} du \tag{30}$$

which for $d = \frac{\lambda}{2}$ reduces to $\left|\sum_{n=0}^{N-1} a_n\right|^2$, obtaining the expression of peak directivity:

$$D_{max} = \frac{\left|\sum_{n=1}^{N} a_n\right|^2}{\sum_{n=0}^{N-1} |a_n|^2} \tag{31}$$

valid for half-wavelength-spaced isotropic elements.

- **Interval Directivity D**

  Using expression (31), we can compute the interval directivity $\mathbf{D}$ as follows:

$$\mathbf{D}_{\max} = \frac{\left|\sum_{n=1}^{N} \mathbf{A}_n\right|^2}{\sum_{n=0}^{N-1} \mathbf{A}_n^2} \tag{32}$$

- **Side Lobe Level for Interval Arrays**

  It's important to define a proper interval version for the significant figure of merit SLL. In our case we can derive from the interval radiation pattern, two different beam patterns:

$$\inf\left\{\mathbf{P}\left(u\right)\right\} = \inf\left\{\left|AF\left(u\right)\right|^2\right\} \tag{33}$$

$$\sup\left\{\mathbf{P}\left(u\right)\right\} = \sup\left\{\left|AF\left(u\right)\right|^2\right\} \tag{34}$$

  Now let us define the interval SLL as $\mathbf{SLL} = \left[\inf\left\{SLL\right\}, \sup\left\{SLL\right\}\right]$ where:

$$\inf\left\{SLL\right\} = -\left[\max_{u \in ML}\left\{\inf\left\{\mathbf{P}\left(u\right)\right\}\right\} - \max_{u \in SL}\left\{\inf\left\{\mathbf{P}\left(u\right)\right\}\right\}\right] \tag{35}$$

$$\sup\left\{SLL\right\} = -\left[\max_{u \in ML}\left\{\inf\left\{\mathbf{P}\left(u\right)\right\}\right\} - \max_{u \in SL}\left\{\sup\left\{\mathbf{P}\left(u\right)\right\}\right\}\right] \tag{36}$$

  where ML and SL are the sets of u corresponding to the Main Lobe and the Side Lobes regions.

- **Pattern Tolerance**

  The Pattern Tolerance parameter $\triangle$ measures the area between the $\sup\left\{\mathbf{P}\left(u\right)\right\}$ and $\inf\left\{\mathbf{P}\left(u\right)\right\}$ and it is defined as:

$$\triangle=\int_{-1}^{1}\left(\sup\left\{\mathbf{P}\left(u\right)\right\}-\inf\left\{\mathbf{P}\left(u\right)\right\}\right)du \tag{37}$$

  A graphical representation of the pattern matching is shown in the following figure.

- **Normalized Pattern Tolerance**

  The Normalized Pattern Tolerance parameter $\triangle_n$ measures the area between the the $\sup\left\{\mathbf{P}\left(u\right)\right\}$ and $\inf\left\{\mathbf{P}\left(u\right)\right\}$ divided by the radiated power $P\left(u\right)$.

$$\Delta_n=\frac{\int_{-1}^{1}\left(\sup\left\{\mathbf{P}\left(u\right)\right\}-\inf\left\{\mathbf{P}\left(u\right)\right\}\right)du}{\int_{-1}^{1}P\left(u\right)du} \tag{38}$$

- **Peak Interval**

  This parameter measure the gap between the infimum and the supremum radiation patterns at the peak of them.

$$\mathbf{P}_{\max}=\left[\inf\left\{\mathbf{P}_{\max}\right\},\,\sup\left\{\mathbf{P}_{\max}\right\}\right] \tag{39}$$

- **Total Tolerance (Amplitudes)**

  This parameter measure the total tolerance considered in a set of excitations amplitudes:

$$T_A=\sum_{n=1}^{N}w\left(\mathbf{A}_n\right) \tag{40}$$

3 Problem Statement and Mathematical Formulation

GOAL: The goal of this work is to analyze the impact of amplitude tolerances on the radiation performances of reconfigurable linear antenna arrays.

Reconfigurable antenna arrays afford more than one service (multi-beam arrays), switching among different feeding networks. Sharing some control points is a good strategy in order to reduce the complexity of the feeding network. Intuitively, in a real scenario, where amplifiers can be affected by manufacturing tolerances, faults and failures, the values assumed by the common control points amplifiers are more critical in beam-forming the desired pattern with respect to the others control points. Exploiting Interval Analysis we can apply an interval over the common control points amplitudes and directly compare the resulting patterns performances.

3.1 Mathematical Formulation

Let us consider a linear array of N elements uniformly spaced along the positive y-axis with the first element located at the center of a reference coordinate system. Using (19) and (20) we can write the array factor of the considered antenna array as:

$$AF(\theta) = \sum_{n=1}^{N} \alpha_n e^{j\Theta_n} \tag{41}$$

where $j = \sqrt{-1}$ is the complex variable and $\Theta_n = (kd(n-1)\sin\theta + \beta_n)$; $n = 1, \ldots, N$, being $k = \frac{2\pi}{\lambda}$ the free-space wavenumber and λ the wavelenght, $\beta_n$is the phase of the $n-th$ excitations, d the inter-element spacing and $\theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$ the angular direction measured from boresight.

Let us now considering a simple reconfigurable antenna array feeding network with P common tail elements and Q independent central elements. The independent central elements are responsible to switch between two different system's services. Let us assume that our antenna array affords sum (Σ) and difference (Δ) beams: the $n-th$ independent element is connected to two different weights, α_n^{Σ} and α_n^{Δ}, for the $\Sigma - BFN$ and $\Delta - BFN$ respectively, that can be toggled by a switch in order to select the desired beam. We have two set of amplitude weights:

$$I^{\Sigma} = \left\{\alpha_1^{\Sigma}, \alpha_2^{\Sigma}, ..., \alpha_{\frac{P}{2}}^{\Sigma}, \alpha_{\frac{P}{2}+1}^{\Sigma}, \alpha_{\frac{P}{2}+2}^{\Sigma}, ..., \alpha_{\frac{P}{2}+Q}^{\Sigma}, \alpha_{N-\frac{P}{2}+1}^{\Sigma}, \alpha_{N-\frac{P}{2}+2}^{\Sigma}, ..., \alpha_N^{\Sigma}\right\} \tag{42}$$

$$I^{\Delta} = \left\{\alpha_1^{\Delta}, \alpha_2^{\Delta}, ..., \alpha_{\frac{P}{2}}^{\Delta}, \alpha_{\frac{P}{2}+1}^{\Delta}, \alpha_{\frac{P}{2}+2}^{\Delta}, ..., \alpha_{\frac{P}{2}+Q}^{\Delta}, \alpha_{N-\frac{P}{2}+1}^{\Delta}, \alpha_{N-\frac{P}{2}+2}^{\Delta}, ..., \alpha_N^{\Delta}\right\} \tag{43}$$

for the sum and difference beam forming networks respectively, where:

$$\alpha_n^{\Sigma} = \alpha_n^{\Delta} = \alpha_n, \quad for\ n = 1, ..., \frac{P}{2}\ and\ n = N - \frac{P}{2} + 1, ..., N \tag{44}$$

Now lets try to split the array factor formula into two main terms that accounts the contributions of the common and the independent control points. Let us define:

$$AF(\theta)_C = \sum_{n=1}^{P/2} \alpha_n e^{j\Theta_n} + \sum_{n=N-\frac{P}{2}+1}^{N} \alpha_n e^{j\Theta_n} \quad (45)$$

$$AF(\theta)_{NC}^{\Sigma} = \sum_{n=\frac{P}{2}+1}^{\frac{P}{2}+Q} \alpha_n^{\Sigma} e^{j\Theta_n} \quad (46)$$

$$AF(\theta)_{NC}^{\Delta} = \sum_{n=\frac{P}{2}+1}^{\frac{P}{2}+Q} \alpha_n^{\Delta} e^{j\Theta_n} \quad (47)$$

where $AF(\theta)_C$ is the term that accounts the contributions of the common elements, while $AF(\theta)_{NC}^{\Sigma}$ and $AF(\theta)_{NC}^{\Delta}$ account the contributions of the independent elements for the sum and difference beams respectively.

We can write two different array factor formulas for the sum ($AF(\theta)^{\Sigma}$) and difference ($AF(\theta)^{\Delta}$) beams:

$$AF(\theta)^{\Sigma} = AF(\theta)_C + AF(\theta)_{NC}^{\Sigma} \quad (48)$$

$$AF(\theta)^{\Delta} = AF(\theta)_C + AF(\theta)_{NC}^{\Delta} \quad (49)$$

Step 1: Synthesis (CP)

The first step is the synthesis of a reconfigurable antenna array fixing the number of common points. A Convex Programming (CP) optimization strategy is used.

- Given the number of elements N, and the sum beam power mask M^{Σ}, the optimal synthesis of the sum pattern is performed obtaining the optimal amplitudes $\alpha_n^{\Sigma},\ for\ n = 1, .., N$;
- then P common amplitudes are fixed, usually the ones placed on the tails;
- given the difference beam power mask M^{Δ}, the optimal synthesis of the difference pattern is performed, optimizing the remaining $Q = N - P$ amplitudes, obtaining the difference pattern's excitations amplitudes $\alpha_n^{\Delta},\ for\ n = 1, .., N$.

Step 2: Analysis (IA)

The two synthesized patterns are then analyzed through IA considering intervals over the excitations amplitudes, in order to evaluate the interval pattern's descriptors, say $\mathbf{SLL}$, $\mathbf{BW}$, $\mathbf{D}$ and Δ.

In order to compare the robustness of the common elements with the robustness of the not-common elements, we are going to fix for each test case an amount T of total tolerance, defined in (40).

The tolerance T is then firstly spread over the common elements amplitudes and the interval patterns' performances are then computed; then the same is done over the remaining elements.

To measure the robustness of a configuration affected by tolerances over the amplitudes excitations, we can measure the distance between the superior bound of the power pattern and the power mask as:

$$\phi(u) = \sup\{\mathbf{P}(u)\} - M(u) \quad (50)$$

We can also define the maximum distance as:

$$\phi_{\max}(u) = \max_{u}\{\sup\{\mathbf{P}(u)\} - M(u)\} \quad (51)$$

Common / Not-common Faulty Elements Robustness Analysis

In the following a comparison between the reconfigurable antenna array's interval descriptors in the presence of **common or not-common** faulty elements is presented.

A linear array composed by $N = 20$ elements, equally spaced by $d = \frac{\lambda}{2}$, affording Σ and Δ beams is considered with:

- P elements shared by both the Σ and Δ feeding networks;
- Q not-common elements for each beam forming network;
- F faulty elements.

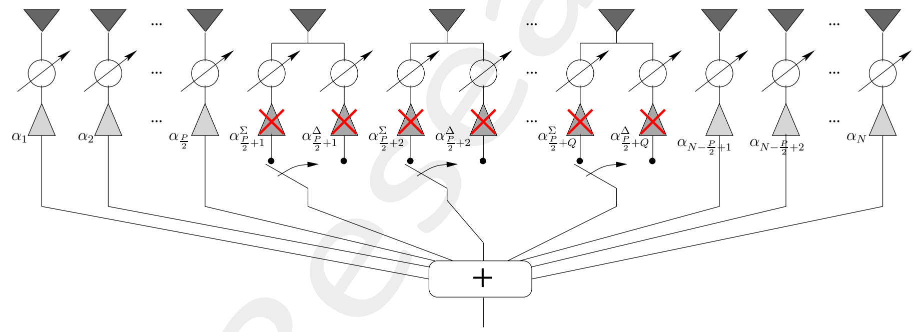

Figure 1. Reconfigurable antenna array feeding network with P common elements and F faulty elements.

In order to model manufacturing errors, faults and failures, of the beam forming network's amplifiers, maximum widths intervals are used, that is, considering normalized amplitudes, the common/not-common interval amplitude of the faulty amplifier is equal to:

$$\mathbf{A}_n = [0.0,\ 1.0] \quad (52)$$

Several reconfigurable antenna arrays with different number of common control points are analyzed:

- $P = 2,\ 4,\ 6,\ 8.$

For each test case, we consider F faulty elements, i.e. intervals are applied to :

- $F = P$ common amplitudes ($\mathbf{A}_n^C$) considering a total tolerance $T_A = P$;
- $F = Q$ not-common amplitudes ($\mathbf{A}_n^{NC}$) considering a $T_A = P$.

For both cases intervals of main array's performances descriptors (SLL, BW, D) are computed.

4 Common / Not-common Faulty Elements Robustness Analysis

4.1 Common Elements $P = 2$

Array Parameters:

- Number Elements: $N = 20$
- Services: Sum / Difference Beams
- Number of Common Elements: $P = 2$
- Element Spacing: $\lambda/2$

Constraints:

- Main Sum Lobe Width: $BW^{\Sigma} = 0.24u$
- Main Difference Lobe Width: $BW^{\Delta} = 0.38u$

Simulation Parameters:

- Sample Points: 2001
- Max Function Evaluations: 6000
- Max Iterations Number: 1000
- Function Tolerance: 1.0×10^{-8}
- Constraint Tolerance: 1×10^{-8}

Algorithm Behaviour:

- Simulation Time Pattern: 20 sec.

In the following figures are reported, for each iteration, the max values evaluated by the objective function and by the constraint function for the sum and difference pattern synthesis.

Sum Beam:

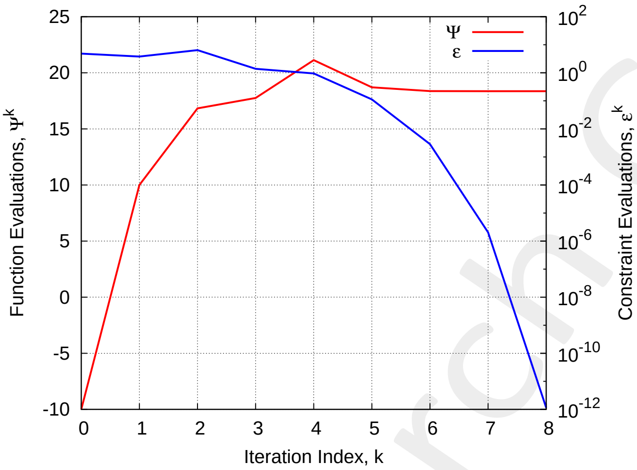

Figure 1. Sum Beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
18.36	1.1×10^{-12}

Table 1. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Difference Beam:

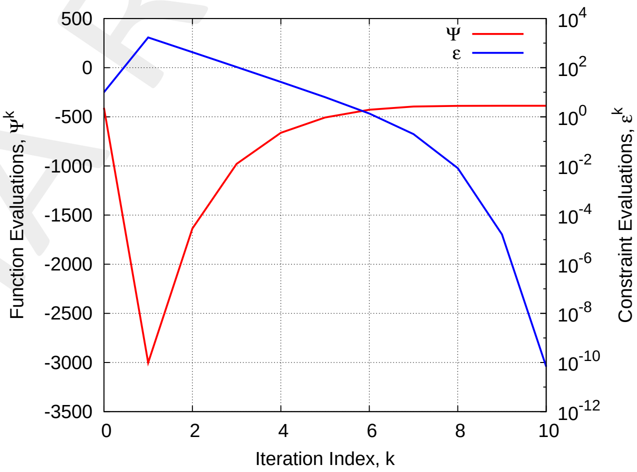

Figure 2. Difference beam optimization's fitness

$max\{\psi(k)\}$	$min\{\varepsilon(k)\}$
-387.9	6.8×10^{-11}

Table 2. Max. value evaluated by ψ; min value evaluated by ε; simulation time.

Excitations:

- Number of Common Elements: $P = 2$

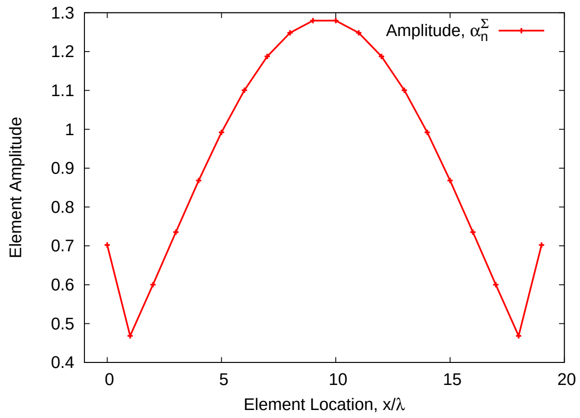

Figure 3. Sum beam's excitations amplitudes

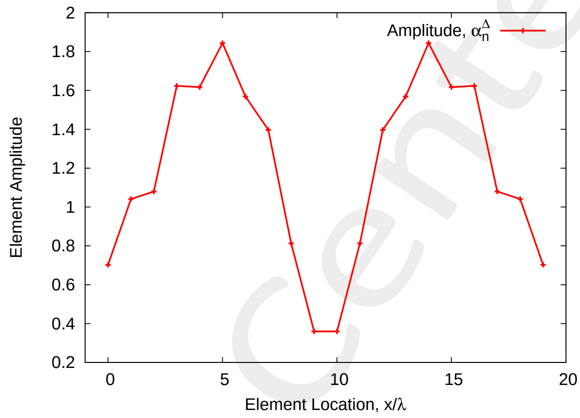

Figure 4. Difference beam's excitations amplitudes

Normalized Excitations:

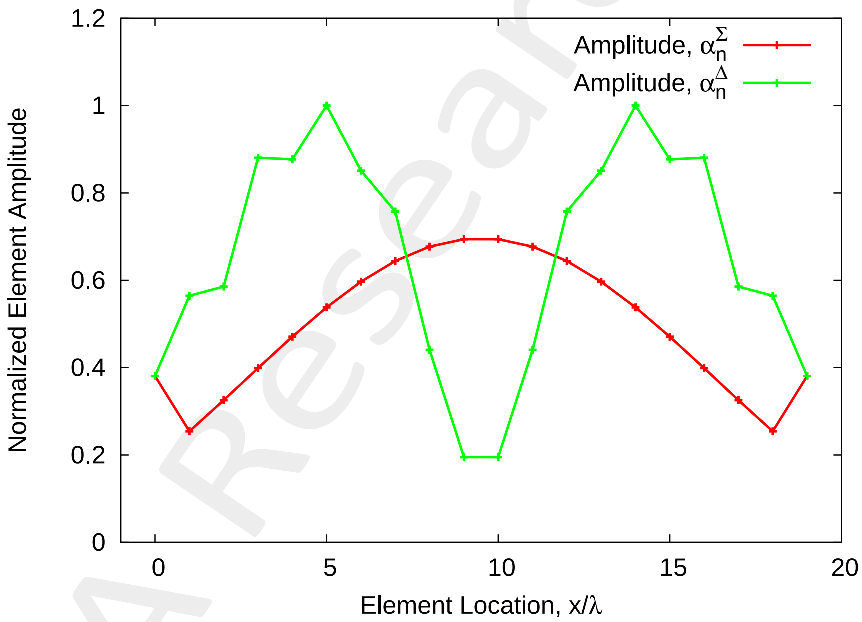

Figure 5. Sum and Difference beam's normalized excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
α_n^Σ	0.3808	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
α_n^Δ	0.3808	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195

Table 3. Sum and Difference beam's nominal amplitudes values (symmetric excitations)

Sum Pattern:

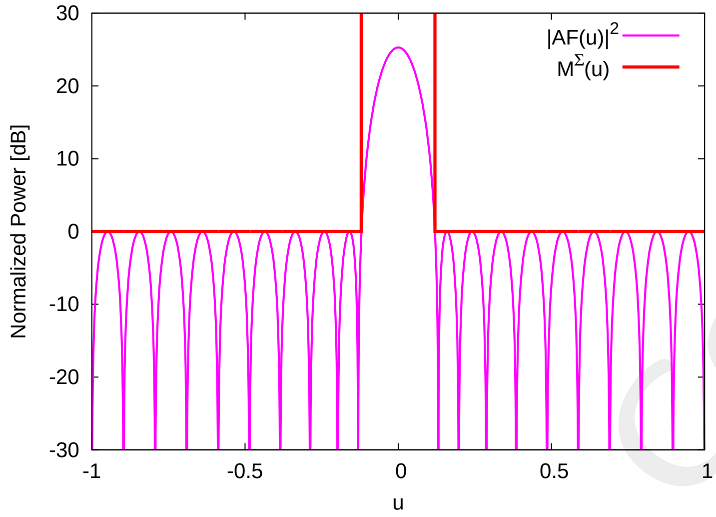

Figure 6. Synthesized Sum Pattern

SLL^{nom} [dB]	$D_{\max}^{nom}$ [dB]	BW^{nom} [u]	ψ_1 [u]
-25.28	12.65	0.104	0.131

Table 4. Sum beam's features values

Difference Pattern:

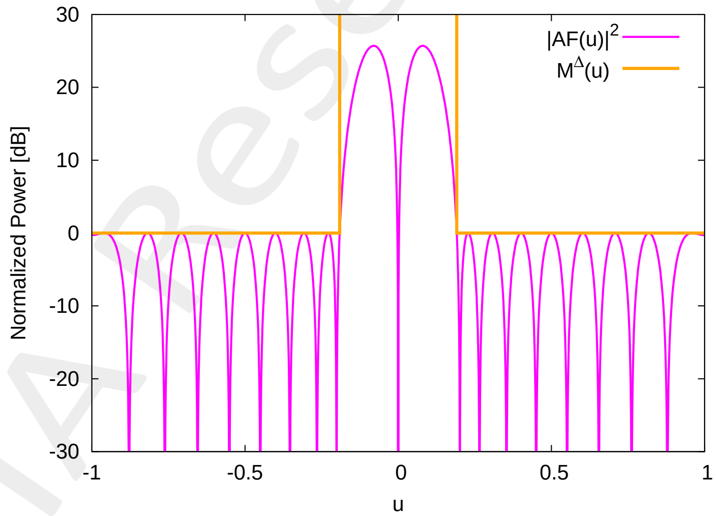

Figure 7. Synthesized Difference Pattern

SLL^{nom} [dB]	K^{nom} [1/rad]	BW^{nom} [u]	ψ_1 [u]
-25.7	1.1281	0.087	0.201

Table 5. Difference beam's features values

Tolerance Over Common Elements - $F = 2$

- Total tolerance : $T = 2$

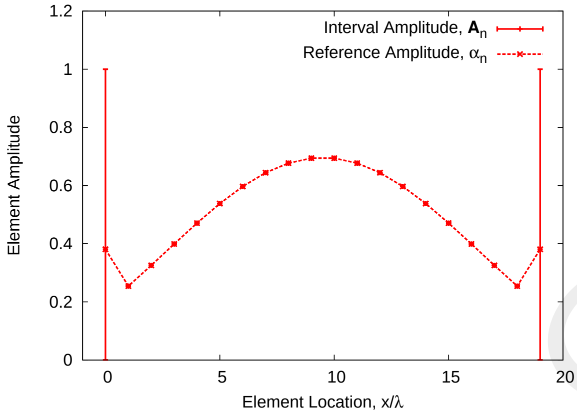

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.3808	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
$\inf\{\mathbf{A}_n\}$	0.0	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
$\sup\{\mathbf{A}_n\}$	1.0	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
$w\{\mathbf{A}_n\}$	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

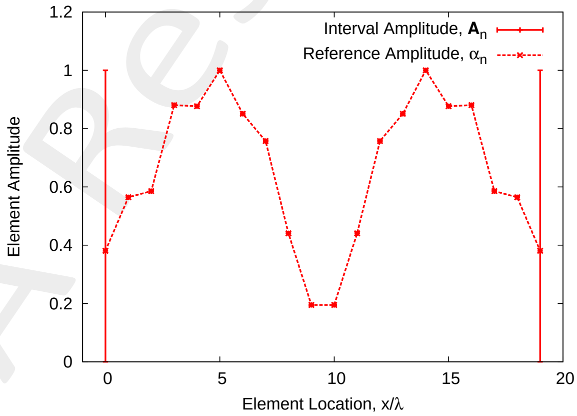

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.3808	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195
$\inf\{\mathbf{A}_n\}$	0.0	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195
$\sup\{\mathbf{A}_n\}$	1.0	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195
$w\{\mathbf{A}_n\}$	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

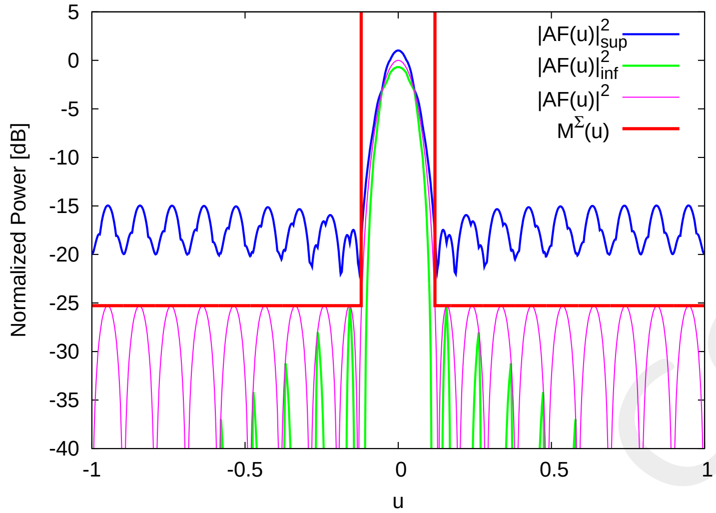

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.073	0.364
inf	-26.37	0.07	10.86	-0.69		
sup	-14.26	0.122	13.83	1.02		

Table 8. Sum Pattern Features

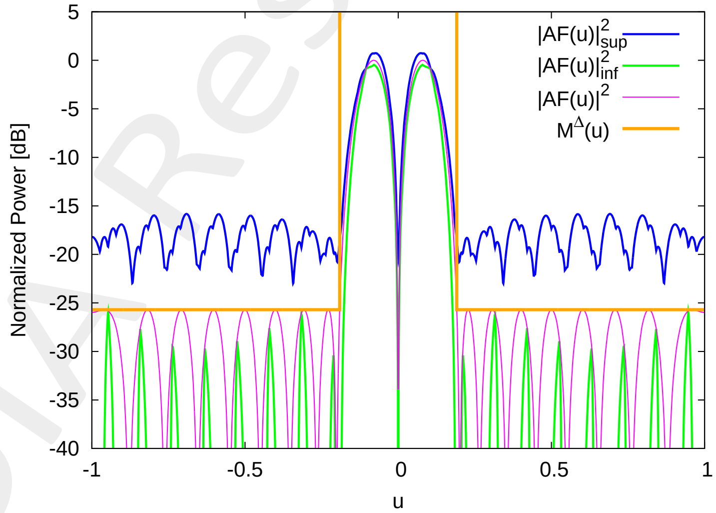

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.7	0.087	1.1281	0.0	0.091	0.317
inf	-26.7	0.067	0.9282	-0.46		
sup	-15.37	0.105	1.3464	0.74		

Table 9. Difference Pattern Features

Tolerance Over Not-Common Elements - $F = 18$

- Total tolerance : $T = 2$

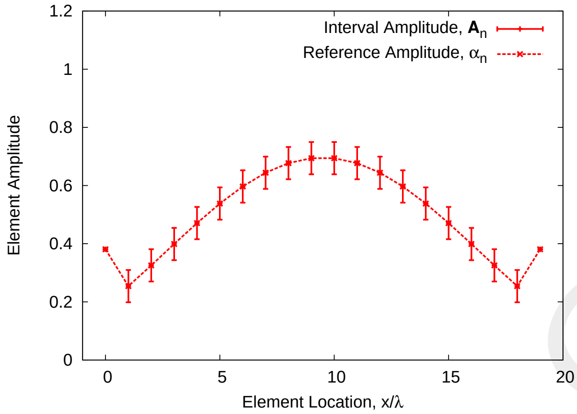

Figure 8. Sum beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.3808	0.2541	0.3254	0.3988	0.4709	0.5381	0.5968	0.644	0.6771	0.6941
$\inf\{\mathbf{A}_n\}$	0.3808	0.1985	0.2699	0.3433	0.4154	0.4825	0.5413	0.5885	0.6215	0.6385
$\sup\{\mathbf{A}_n\}$	0.3808	0.3096	0.381	0.4544	0.5265	0.5936	0.6524	0.6996	0.7326	0.7496
$w\{\mathbf{A}_n\}$	0.0	0.1112	0.1112	0.1112	0.1112	0.1112	0.1112	0.1112	0.1112	0.1112

Table 6. Sum beam's nominal and interval amplitudes values (symmetric excitations)

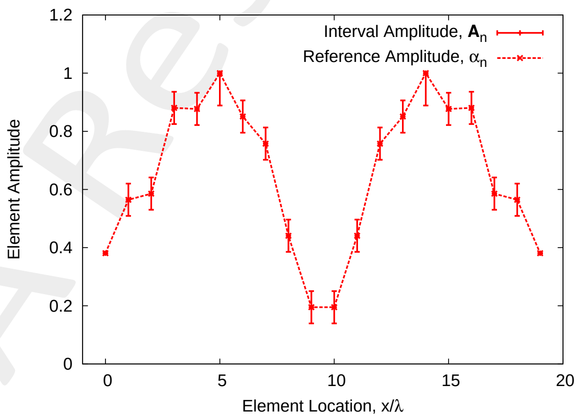

Figure 9. Difference beam's excitations amplitudes

n	1	2	3	4	5	6	7	8	9	10
nominal	0.3808	0.5646	0.5856	0.8806	0.877	1.0	0.8509	0.7576	0.4408	0.195
$\inf\{\mathbf{A}_n\}$	0.3808	0.509	0.5301	0.8251	0.8214	0.8889	0.7953	0.702	0.3853	0.1394
$\sup\{\mathbf{A}_n\}$	0.3808	0.6201	0.6412	0.9362	0.9326	1.0	0.9064	0.8132	0.4964	0.2505
$w\{\mathbf{A}_n\}$	0.0	0.1112	0.1112	0.1112	0.1112	0.1112	0.1112	0.1112	0.1112	0.1112

Table 7. Difference beam's nominal and interval amplitudes values (symmetric excitations)

Sum / Difference Interval Patterns:

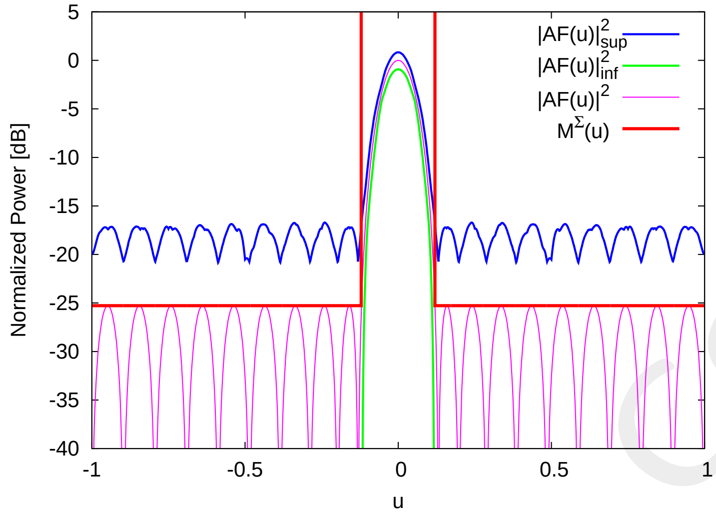

Figure 10. Synthesized Sum Pattern

	SLL [dB]	**BW** [u]	$\mathbf{D}_{\max}$ [dB]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.28	0.104	12.65	0.0	0.077	0.38
inf	$-\infty$	0.068	11.05	-0.92		
sup	-15.79	0.132	14.26	0.83		

Table 8. Sum Pattern Features

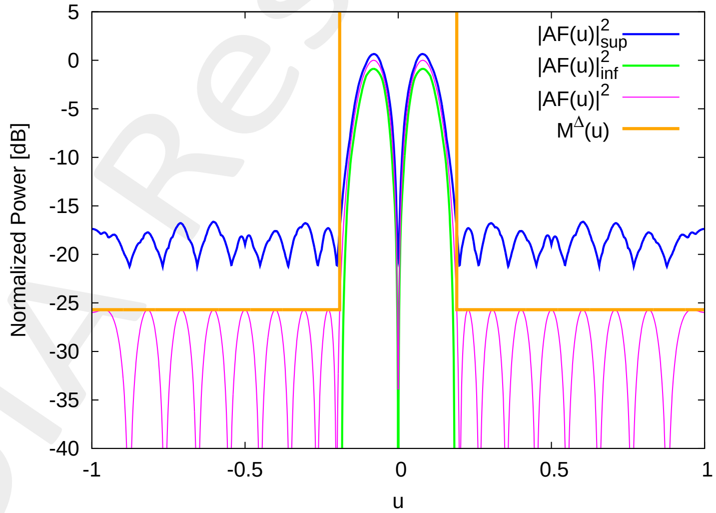

Figure 11. Synthesized Difference Pattern

	SLL [dB]	**BW** [u]	**K** [1/rad]	$\mathbf{P}_{\max}$ [dB]	Δ	Δ_n
nominal	-25.7	0.087	1.1281	0.0	0.0934	0.3268
inf	$-\infty$	0.064	0.985	-0.88		
sup	-15.76	0.11	1.301	0.65		

Table 9. Difference Pattern Features

More information on the topics of this document can be found in the following list of references.

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