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LATTICES

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# Stochastic Ray Propagation in Stratified Random Lattices

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# Stochastic Ray Propagation in Stratified Random Lattices

Anna Martini, Massimo Franceschetti, and Andrea Massa

## Abstract

Ray propagation in stratified semi-infinite percolation lattices consisting of a succession of different uniform-density layers is considered. Assuming that rays undergo specular reflections on the occupied sites, the propagation depth inside the medium is analytically estimated in terms of the probability that a ray reaches a prescribed level before being reflected back in the above empty half-plane. Numerical Monte-Carlo-like experiments validate the proposed solution.

## Key words:

Percolation theory, Stochastic ray tracing, Wave propagation, Stratified random media.

# 1 Introduction

In the last years, wave propagation in random media has gained an increasing attention mostly due to the huge amount of practical problems where propagation environments are suitable to be described by stochastic models rather than being deterministically characterized. For instance, let us think about applications arising in the field of wireless communication [1][2][3] and remote sensing (see [4] and the references cited therein).

In such a framework, we study electromagnetic wave propagation in a semi-infinite percolation lattice [5] of square sites, modeling a random distribution of scatterers. The electromagnetic source is assumed to be external to the half-plane and it radiates a monochromatic plane wave impinging on the lattice with a known angle  $\theta$ . Sites are assumed to be large with respect to the wavelength. This allows to model the incident wave in terms of parallel rays. Such rays undergo specular reflection on obstacles, while other electromagnetic interactions are neglected. The aim is estimating the probability that a single ray reaches a prescribed level  $k$  inside the lattice before being reflected back in the above empty half-plane,  $\Pr\{0 \mapsto k \prec 0\}$ .

This problem was addressed for the first time in [1], where the authors considered the case of a uniform percolation lattice, where each cell may be occupied with a known probability  $q$ . To ensure propagation, such occupancy probability is assumed to be lower than a prescribed value  $q_c = 1 - p_c$ ,  $p_c$  being the so-called percolation threshold [5] ( $p_c \approx 0.59275$  for the two-dimensional case). Ray propagation was modeled in terms of a stochastic process defined as the sum of successive ray jumps. The final result was expressed as a combination of two terms: the probability mass function  $\Pr\{r_0 = i\}$  of the first jump  $r_0$  and the probability  $\Pr\{i \mapsto k \prec 0 | r_0 = i\}$  that a ray reaches level  $k$  before escaping in the above empty half-plane given the level where the first reflection occurs. The latter term was estimated by applying the theory of the Martingale random processes [6] and the so called Wald approximation.

Extension of this approach to the inhomogeneous case has been proposed in [7][8], where the scatterers distribution has been described by a one-dimensional obstacles density profile,  $f(j) q(j)$ ,  $j$  being the row index. Numerical experiments and mathematical considerations have shown that the analytical solution holds true in correspondence of obstacles

density profiles with small variations. A little thought shows that this is due to the fact that ray jumps following the first one are considered as a single mathematical entity, i.e.,  $\Pr\{i \mapsto k < 0 \mid r_0 = i\}$ . Thus, such a formulation is not able to faithfully describe  $\Pr\{0 \mapsto k < 0\}$  in correspondence with abrupt variations in the density profile  $q(j)$ .

This letter is aimed at overcoming such a drawback by providing an ad-hoc formulation for describing propagation in stratified random lattices consisting of a succession of different uniform-density layers. The work is organized as follows. In Section 2, the mathematical formulation is presented. Section 3 provides some numerical experiments performed on simple test cases. Final comments and conclusions are drawn in Section 4.

## 2 Problem Statement and Mathematical Formulation

Let us consider a stratified semi-infinite percolating lattice described by the following obstacles density distribution

$$q(j) = \begin{cases} q_1 & l_0 = 0 < j \leq l_1, \\ q_2 & l_1 < j \leq l_2, \\ \vdots & \\ q_n & l_{n-1} < j \leq l_n, \\ \vdots & \end{cases} \quad (1)$$

where  $q(j) = 1 - p(j)$  is the probability that a site is occupied at level  $j$ . In other words, the medium is a succession of layers  $\{L_n; n \geq 1\}$ , each one made up of  $l_n - l_{n-1}$  levels with occupancy probability  $q_n$ . An example of a stratified random lattice with three layers and the relative obstacles density distribution are shown in Figure 1. For the considered configuration, our aim is to find the probability  $\Pr\{0 \mapsto k < 0\}$ .

In each uniform layer belonging to the stratified lattice, the propagation is described through the model proposed in [1]. In particular, the probability that a ray traveling with positive direction in level  $(l_{n-1} + 1)$  reaches level  $l_n$  before being reflected back in level

$(l_{n-1} + 1)$ ,  $P_n \hat{=} \Pr \{(l_{n-1} + 1) \mapsto l_n \prec (l_{n-1} + 1)\}$ , turns out to be [1],

$$P_n = \begin{cases} 1 & l_n = l_{n-1} + 1, \\ \frac{p_n}{q_{e_n} N_n} [1 - p_{e_n}^{N_n}] & l_n > l_{n-1} + 1, \end{cases} \quad (2)$$

where  $p_{e_n} = 1 - q_{e_n} = p_n^{\tan\theta+1}$  is the effective probability a ray freely crosses a level with occupancy probability  $q_n$  and  $N_n = (l_n - l_{n-1} - 1)$ . Numerical experiments and mathematical considerations show that (2) satisfactorily performs for incidence angle  $\theta$  not too far from  $45^\circ$  and for dense propagation media [9].

Now, in order to describe propagation in the whole stratified lattice, the probabilities  $P_n$  of each single layer ( $n \geq 1$ ) must be conveniently combined. If we assume that the level  $k$  belongs to the layer  $L_K$ , i.e.,  $l_{K-1} < k \leq l_K$ ,  $K \geq 1$ , our problem is formally described by the Markov chain [10] depicted in Figure 2, where states  $j^+$  and  $j^-$  denote a ray traveling in level  $j$  with positive and negative direction, respectively, and  $Q_n \hat{=} 1 - P_n$ . With reference to the Markov chain, we state our main result as follows (see Appendix A for a detailed proof)

$$\Pr \{0 \mapsto k \prec 0\} = \frac{p_1}{\frac{1}{P_1} + p_1 \sum_{n=2}^K \left[ \frac{1-P_n}{p_n P_n} + \frac{q_n}{p_n p_{n-1}} \right]}, \quad (3)$$

where  $P_K$  is evaluated according to (2) by replacing  $l_K$  with  $k$ .

An observation is appropriate. When  $K = 1$ , we are dealing with the homogeneous case and equation (3) takes the form

$$\Pr \{0 \mapsto k \prec 0\} = p_1 P_1 = \begin{cases} p_1, & k = 1, \\ \frac{p_1^2 [1 - p_{e_1}^{(k-1)}]}{q_{e_1}^{(k-1)}}, & k > 1. \end{cases} \quad (4)$$

Such a result is a slightly different version of that in [1], since it takes into account that a ray traveling with negative direction inside level 1 surely escapes from the grid because there are not any occupied horizontal faces between level 1 and level 0. As a matter of fact, the approach in [1] is fine in evaluating provided that  $x$  and  $y$ , and the levels between them, have the same occupancy probability because  $\Pr \{i \mapsto k \prec 0 \mid r_0 = i\}$  is

estimated on the basis of a distance criterion. Consequently, since in our configuration level 0 is empty, we can not directly apply [1] for computing  $\Pr\{0 \mapsto k \prec 0\}$ . Therefore,  $\Pr\{0 \mapsto k \prec 0\}$  is evaluated as the product of two terms, the probability  $p_1$  to enter the first level and the probability  $P_1 = \Pr\{x \mapsto y \prec x\}_{x=1, y=k}$  computed as in [1].

In passing and as expected, it can be noticed that (3) does not reduce to (4) in the limit case when  $p_n = p$ ,  $n = 1, \dots, K$  since (3) is not an extension of the result in [1] (where the ray jumps following the first one are evaluated through an approximation on the basis of a distance criterion), but it is obtained by mathematically binding in the Markov chain depicted in Figure 2 the results concerned with the uniform case.

### 3 Numerical Validation

In order to validate the proposed solution, an exhaustive set of numerical experiments has been carried out taking into account two-, three- and four-layers scenarios. In the following, the results of selected representative test cases are reported. For comparison purposes, the propagation depth has been estimated in the first  $K = 32$  levels by Monte-Carlo-like ray-tracing experiments by following the procedure detailed in [1].

In order to estimate the effectiveness of the proposed model, let us define the following error indexes, namely the prediction error  $\delta_k$

$$\delta_k \triangleq \frac{|\Pr_R\{0 \mapsto k\} - \Pr_P\{0 \mapsto k\}|}{\max_k [\Pr_R\{0 \mapsto k\}]} \times 100, \quad (5)$$

and the mean error  $\langle \delta \rangle$

$$\langle \delta \rangle \triangleq \frac{1}{K_{max}} \sum_{k=1}^{K_{max}} \delta_k, \quad (6)$$

where the sub-scripts  $R$  and  $P$  indicate the values estimated with the reference approach and through (3), respectively.

Firstly, we fix the incidence angle,  $\theta = 45^\circ$ , and we analyze how the obstacles density at each layer and the size of the variation in the occupation probability value between adjacent layers, namely  $S_{n,n+1} = |q_n - q_{n+1}|$ , affect the performances. With reference to Table I, where mean error values relative to single-step profiles are reported, it can be observed that the effectiveness of the proposed solution does not depend on  $S_{1,2}$ . As

an example, let us consider single-step profiles having  $q_1 = 0.35$  (last row in Tab. I). The matching between reference data and the reconstruction obtained by means of (3) is good whatever  $S_{1,2}$  and it is comparable with that of the uniform case. The capability of the proposed approach in carefully modeling the behavior of  $\Pr\{0 \mapsto k \prec 0\}$  is also evident for other single-step profiles as confirmed by the values of the error index (Tab. I). For a fixed value of  $q_1$ , the mean error decreases when  $q_2$  increases, independently from  $S_{1,2}$ . Such an event points out that the prediction accuracy is affected only by the obstacles density at each layer. In particular, more dense the layers are, lower the mean error is. This behaviour is fully predictable, since the accuracy of (2) - the building block in deriving the final result - increases when the occupancy probability value tends to the percolation threshold [9]. Such a trend, verified for single-step profiles, is further confirmed when random lattices with a higher number of layers are taken into account. With reference to Figure 3, where plots of  $\Pr\{0 \mapsto k \prec 0\}$  for three-layers profiles with fixed  $q_1 = q_3 = 0.15$  are reported, it can be observed that matching between reference and estimated data gets better for higher  $q_2$ , although  $S_{n,n+1}$  increases ( $S_{1,2} = S_{2,3} = 0.1$  and  $S_{1,2} = S_{2,3} = 0.2$  when  $q_2 = 0.05$  and  $q_2 = 0.35$ , respectively). This is confirmed by the mean error values ( $\langle\delta\rangle_{q_2=0.05} = 3.13\%$  vs.  $\langle\delta\rangle_{q_2=0.35} = 0.8\%$ ). In Figure 4, we compare the results relative to two four-layers profiles, the former very sparse ( $q_1 = q_3 = 0.15$  and  $q_2 = q_4 = 0.05$ ) and the latter very dense ( $q_1 = q_3 = 0.35$  and  $q_2 = q_4 = 0.25$ ). As expected, we get better performances for the more dense profile, as confirmed by the mean error values ( $\langle\delta\rangle = 2.98\%$  vs.  $\langle\delta\rangle = 0.7\%$ ).

Now, the effects of the incidence angle  $\theta$  on the performances of (3) are analyzed. Towards this end, let us define the global mean error  $\Delta$ ,

$$\Delta \triangleq \frac{1}{\Gamma} \sum_{s=1}^{\Gamma} \langle\delta\rangle_s, \quad (7)$$

$\Gamma$  being the total number of scenarios and  $\langle\delta\rangle_s$  the mean error relative to the  $s$ -th distribution. Figure 5 plots  $\Delta$  obtained by considering the whole set of three- and four-layers configurations that can be built by varying the occupation probability of each layer  $\{q_n; n = 1, \dots, K\}$  between 0.05 and 0.35 with a step of 0.1. As expected [9], we observe that in both cases results get worse as  $\theta$  diverges from  $45^\circ$ . In particular,  $\Delta$

ranges from  $\{\Delta\}_{\theta=45^\circ} = 1.35\%$  up to  $\{\Delta\}_{\theta=15^\circ} = 5.52\%$  and from  $\{\Delta\}_{\theta=45^\circ} = 1.28\%$  up to  $\{\Delta\}_{\theta=15^\circ} = 5.54\%$  for three- and four-layers profiles, respectively. Moreover, it is interesting to observe that the plots concerned with three- and four-layers scenarios almost overlap. Such an event indicates that, on average, the accuracy of the approach is not affected by the number of layers taken into account.

## 4 Conclusions

Ray propagation in stratified half-plane random lattices illuminated by a monochromatic plane wave that undergoes specular reflections on the occupied sites has been studied. We have estimated the penetration depth by mathematically binding in a Markov chain results relative to uniform lattices [1], thus overcoming the limits of the solution presented in [7] when dealing with stratified profiles [9].

The proposed approach has been validated by means of computer-based ray-tracing experiments showing that the proposed solution satisfactorily performs in describing the behavior of  $\Pr\{0 \mapsto k \prec 0\}$  when abrupt variations in the obstacles density profiles occur. As a matter of fact, the prediction accuracy is affected neither by the size of the density variation nor by the number of such variations (i.e., the number of layers of the lattice). On the other hand, the same limitations of the solution relative to the uniform case [9], still remain (i.e., better predictions turn out in correspondence with dense media and incidence angles near to  $45^\circ$ ).

# Appendix A

In this Section, we prove (3) by induction.

The case  $K = 1$  has been discussed at the end of Section 2. Thus, we need to show that (3) holds true if it holds for  $(K - 1)$ . Towards this end, by making reference to the Markov chain depicted in Figure 2, we express  $\Pr\{0 \mapsto k \prec 0\}$  as the product of three terms

$$\Pr\{0 \mapsto k \prec 0\} = \Pr\{A\} \Pr\{B\} \Pr\{C\} \quad (8)$$

where

$$\Pr\{A\} = \Pr\{0^+ \mapsto l_{K-1}^+ \prec 0^-\}, \quad (9)$$

$$\Pr\{B\} = \Pr\{l_{K-1}^+ \mapsto (l_{K-1} + 1)^+ \prec 0^-\}, \quad (10)$$

$$\Pr\{C\} = \Pr\{(l_{K-1} + 1)^+ \mapsto k \prec 0^-\}. \quad (11)$$

Let us consider  $\Pr\{C\}$ . By observing the Markov chain, we have

$$\Pr\{C\} = P_K + Q_K \Pr\{(l_{K-1} + 1)^- \mapsto (l_{K-1} + 1)^+ \prec 0^-\} \Pr\{C\}, \quad (12)$$

and accordingly,

$$\begin{aligned} \Pr\{C\} &= \frac{P_K}{1 - Q_K \Pr\{(l_{K-1} + 1)^- \mapsto (l_{K-1} + 1)^+ \prec 0^-\}} \\ &= \frac{P_K}{P_K + Q_K \Pr\{(l_{K-1} + 1)^- \mapsto 0^- \prec (l_{K-1} + 1)^+\}}, \end{aligned} \quad (13)$$

the last equality following from mutual exclusivity. Now, it can be proved [8] that, whatever level  $j$  inside the lattice we are considering,

$$\Pr\{j^- \mapsto 0^- \prec j^+\} = \frac{\Pr\{0^+ \mapsto j^+ \prec 0^-\}}{p(j)}, \quad (14)$$

$p(j)$  being the probability a site is free at level  $j$ , and accordingly

$$\begin{aligned} \Pr\{C\} &= \frac{p_K P_K}{p_K P_K + Q_K \Pr\{0^+ \mapsto (l_{K-1} + 1)^+ \prec 0^-\}} \\ &= \frac{p_K P_K}{p_K P_K + Q_K \Pr\{A\} \Pr\{B\}}. \end{aligned} \quad (15)$$

As far as  $\Pr\{B\} = \Pr\{l_{K-1}^+ \mapsto (l_{K-1} + 1)^+ \prec 0^-\}$  is concerned, by following similar reasoning as in getting  $\Pr\{C\}$ , we obtain

$$\begin{aligned} \Pr\{B\} &= p_K + q_K \Pr\{l_{K-1}^- \mapsto l_{K-1}^+ \prec 0^-\} \Pr\{B\} = \\ &= p_K + q_K \left[1 - \frac{\Pr\{A\}}{p_{K-1}}\right] \Pr\{B\} \end{aligned} \quad (16)$$

and thus,

$$\Pr\{B\} = \frac{p_{K-1}p_K}{p_{K-1}p_K + q_K \Pr\{A\}}. \quad (17)$$

By applying to (8), (15), and (17), after some algebra we have

$$\Pr\{0 \mapsto k \prec 0\} = \frac{1}{\frac{1}{\Pr\{A\}} + \frac{1-p_K}{p_K p_K} + \frac{q_K}{p_{K-1}p_K}} \quad (18)$$

Now, our main result (3) holds true for whatever  $k$  belonging to layer  $L_{K-1}$ . Thus, it holds true also for  $k = l_{K-1}$  and accordingly we can write

$$\Pr\{A\} = \frac{p_1}{\frac{1}{p_1} + p_1 \sum_{n=2}^{K-1} \left[ \frac{1-p_n}{p_n p_n} + \frac{q_n}{p_n p_{n-1}} \right]}. \quad (19)$$

By substituting (19) into (18), we simply get our final result (3).

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## Figure Captions

- **Figure 1.** Sketch of ray propagation in a three layers random lattice (left-hand side) and the obstacles density distribution relative to the grid (right-hand side).
- **Figure 2.** Markov chain modeling ray propagation towards level  $k$ .
- **Figure 3.** Three-layers obstacles density profile with  $l_1 = 8$ ,  $l_2 = 16$  and  $q_1 = q_3 = 0.15$  - Estimated values of  $Pr\{0 \mapsto k \prec 0\}$  versus  $k$  when  $\theta = 45^\circ$  for (a)  $q_2 = 0.05$  and (b)  $q_2 = 0.35$ . Crosses denote reference data, while solid line describes the prediction obtained by (3).
- **Figure 4.** Four-layers obstacles density profiles with  $l_1 = 8$ ,  $l_2 = 16$  and  $l_3 = 24$  - Estimated values of  $Pr\{0 \mapsto k \prec 0\}$  versus  $k$  when  $\theta = 45^\circ$  for (a) a sparse profile ( $q_1 = q_3 = 0.15$  and  $q_2 = q_4 = 0.05$ ) and (b) a dense profile ( $q_1 = q_3 = 0.35$  and  $q_2 = q_4 = 0.25$ ). Crosses denote reference data, while solid line describes the prediction obtained by (3).
- **Figure 5.** Three-layers obstacles density profiles ( $l_1 = 8$  and  $l_2 = 16$ ) and four-layers obstacles density profiles ( $l_1 = 8$ ,  $l_2 = 16$  and  $l_3 = 24$ ) - Global mean error  $\Delta$  versus the incidence angle  $\theta$ .

## Table Captions

- Table I. Step profile - Mean error  $\langle\varphi\rangle$  for different values of  $q_1$  and  $q_2$  when  $\theta = 45^\circ$ . For completeness, values relative to uniform configurations obtained by (4) are reported in square brackets on the diagonal.

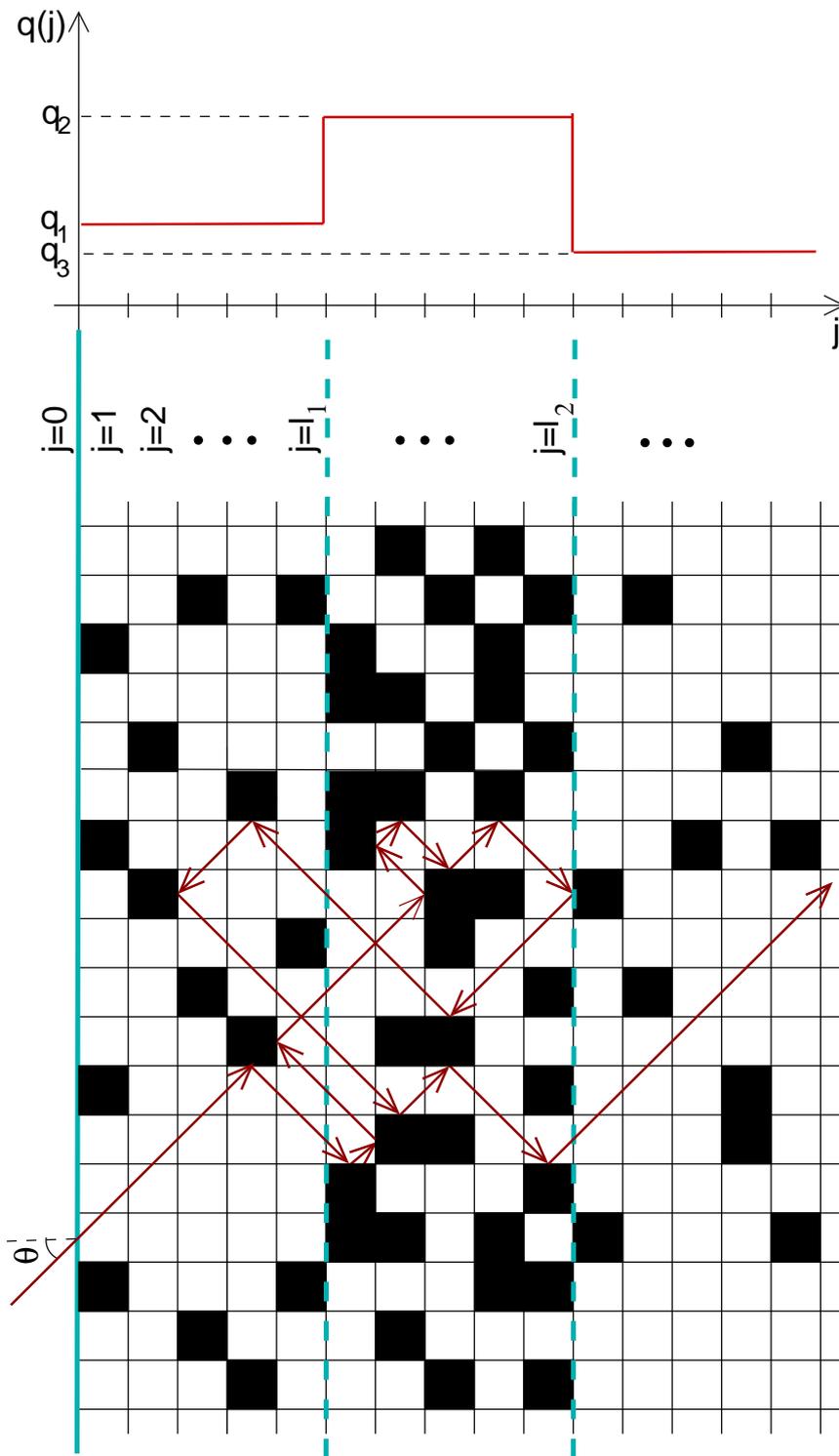
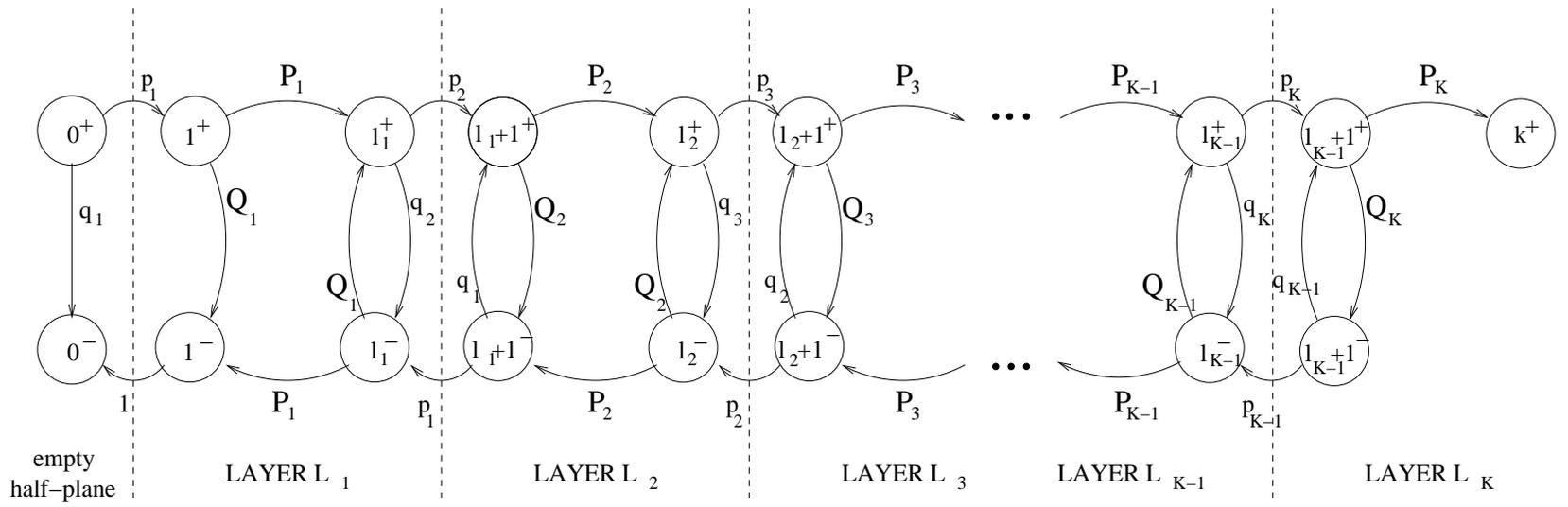
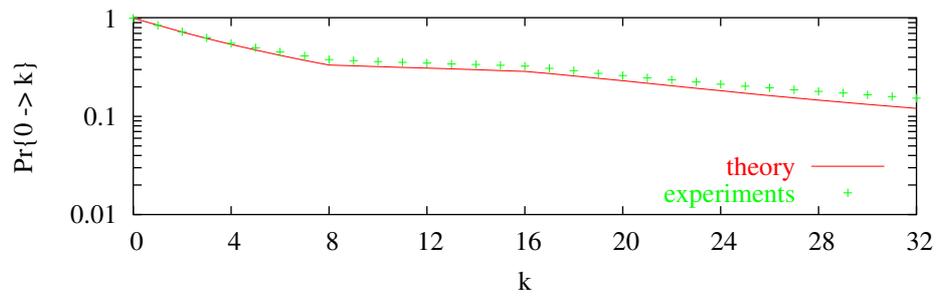


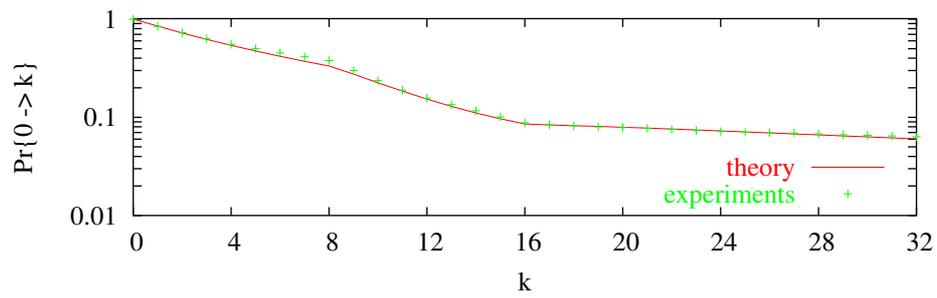
Fig. 1 - A. Martini *et al.*, "Stochastic Ray Propagation ..."

Fig. 2 - A. Martini *et al.*, "Stochastic Ray Propagation ..."



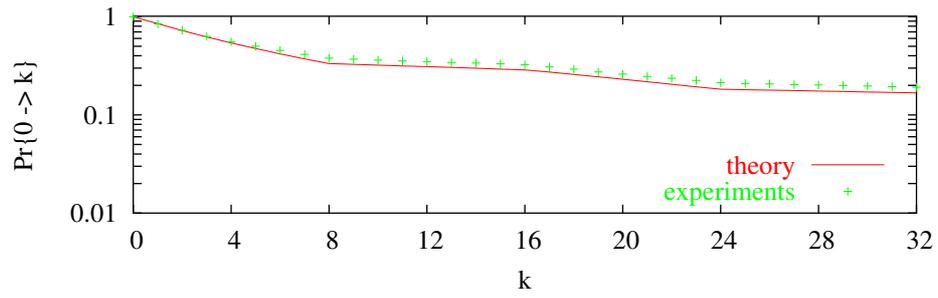


(a)

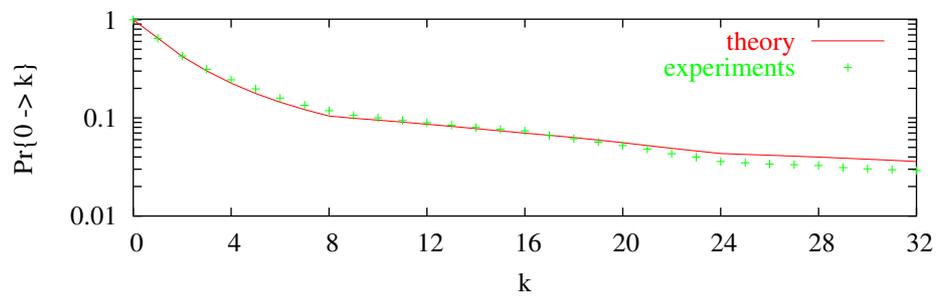


(b)

Fig. 3 - A. Martini *et al.*, "Stochastic Ray Propagation ..."



(a)



(b)

Fig. 4 - A. Martini *et al.*, "Stochastic Ray Propagation ..."

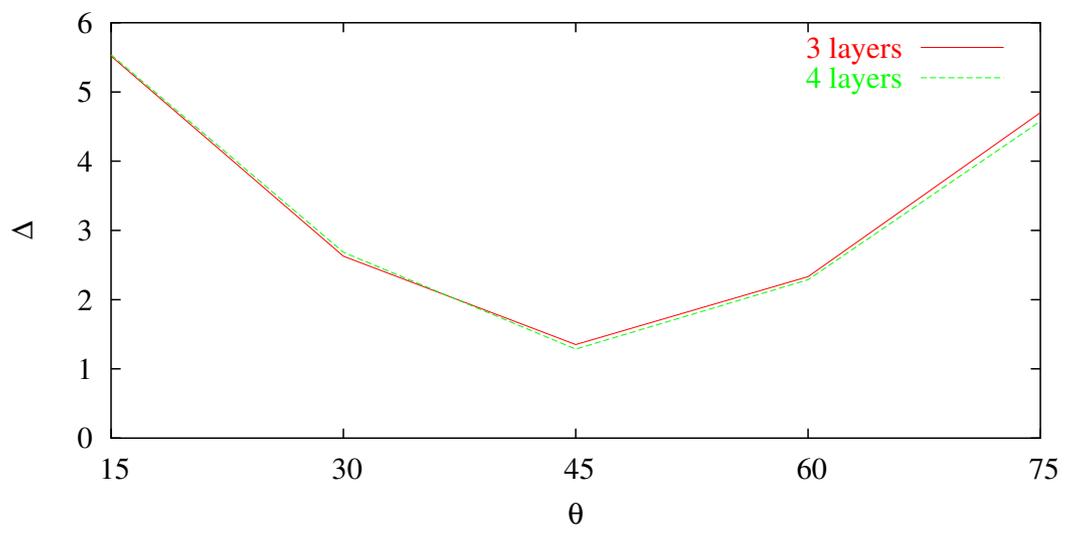


Fig. 5 - A. Martini *et al.*, "Stochastic Ray Propagation ..."

$q_1 \setminus q_2$	0.05	0.15	0.25	0.35
0.05	[3.87]	2.83	1.64	0.53
0.15	3.06	[4.09]	1.86	0.88
0.25	2.31	2.10	[1.93]	0.81
0.35	0.47	0.50	0.31	[0.56]

Tab. I - A. Martini *et al.*, "Stochastic Ray Propagation ..."