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GENETICALLY-DESIGNED ARBITRARY LENGTH ALMOST  
DIFFERENCE SETS

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# Genetically-Designed Arbitrary Length Almost Difference Sets

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Almost Difference Sets (ADSs) have important applications in cryptography, coding theory, and antenna array thinning. In this letter, a new approach is proposed to derive ADSs of arbitrary lengths. Such a technique recasts the ADS design as a combinatorial optimization problem successively solved by means of a suitable binary Genetic Algorithm. New ADSs are derived to assess the effectiveness of the proposed approach.

*Introduction:* In recent years, the design of binary sequences with three-level autocorrelation [e.g., the so-called Almost Difference Sets (ADSs)] has gained considerable attention because of several applications in cryptography [1], coding [2]-[4], and antenna array synthesis [5]. Several construction techniques have been already developed [6]-[9], and even large repositories are now available [10]. However, the fact that ADS sequences of arbitrary length are (at present) not available [6]-[10] is a limitation for their use in real-world problems. As a matter of fact, since ADS synthesis techniques are usually based on the *cyclotomy* property [9], they generate sequences characterized by specific cyclotomic numbers and not with arbitrary length.

In this letter, a new method is proposed for the synthesis of ADS sequences of arbitrary length. The approach reformulates the ADS design in terms of a combinatorial optimization problem where the cost function quantifies the misfit between the autocorrelation of a binary sequence and the three-valued function of the ADSs. A binary genetic algorithm (GA) [11][12] is used to minimize such a cost function because of its “hill climbing” features and its ability to sample in a

very effective fashion the binary solution space [11]. Representative numerical results are analyzed to give some indications on the effectiveness of the proposed approach and explicit expressions of new ADS sequences of different lengths are derived.

*ADS Synthesis as a Combinatorial Optimization Problem:* Let us consider a binary sequence  $\underline{b} = \{b(n); n = 0, \dots, N-1\}$  of length  $N$  whose cyclic autocorrelation function is defined as

$$A(\tau) = \sum_{n=1}^N b(n)b[(n+\tau) \bmod N] \quad (1)$$

where  $\tau = 0, \dots, N-1$  and  $b(n) \in \{0,1\}$ .

A  $(N, K, \Lambda, t)$ -ADS is a binary sequence characterized by an autocorrelation function [6]-[9] given by

$$A_{ADS}(\tau) = \begin{cases} K & \tau = 0 \\ \Lambda & \text{for } t \text{ values of } \tau. \\ \Lambda + 1 & \text{otherwise} \end{cases} \quad (2)$$

In order to determine an ADS of arbitrary length  $N$ , a suitable optimization problem is formulated and then solved by a means of a GA-based iterative procedure [11][12]. Starting from an initial ( $i = 1$ ) population composed by a set of  $P$  randomly-generated binary sequences,  $\underline{b}_p^i$ ,  $p = 0, \dots, P-1$ , the trial solutions iteratively ( $i$  being the iteration index) undergo the genetic evolution through selection, crossover (with probability  $\chi$ ), and mutation (with probability  $\mu$ ) until a suitable solution  $\underline{b}_{opt}$  is found. The degree of optimality of each trial sequence is evaluated by means of the following fitness function

$$F(\underline{b}_p^i) = \alpha(L_p^i - 3) + \beta R_p^i \quad (3)$$

where  $L_p^i$  is the number of levels of the autocorrelation function of  $\underline{b}_p^i$  and  $R_p^i$  is the number of  $\tau$  values for which  $A_p^i(\tau)$  differs from (2). Moreover,  $\alpha$  and  $\beta$  are user-defined real weights. The iterations stop when either an ADS sequence is found [i.e.,  $F(\underline{b}_{opt}) = 0$ ] or the maximum number of iterations  $I$  is reached ( $i = I$ ). It is worth to notice that, since (3) measures the “similarity” between  $A_p^i(\tau)$  and the autocorrelation function of the ADSs, the optimization technique synthesizes the optimal binary sequences by iteratively approaching the desired three-level autocorrelation, thus avoiding the constraints on the sequence length of state-of-the-art generation methods [6]-[9].

*Numerical validation:* For a preliminary assessment, the following parameter setup has been chosen throughout the numerical validation:  $\chi = 0.9$ ,  $\mu = 0.01$ ,  $\alpha = 10^{-2}$ ,  $\beta = 10^{-4}$ , and  $P = N$ .

The first example deals with the synthesis of an ADS of length  $N = 24$ . Although the dimension of the solution space is not negligible ( $U = 2^{24} \cong 1.7 \times 10^7$ ,  $U$  being the number of binary sequences), very few iterations ( $i_{opt} = 8$ ) are enough to reach a binary sequence with a three-level autocorrelation [Fig. 1(a)], which (to the best of authors’ knowledge) corresponds to a new ADS, namely the (24,8,2,13)-ADS in Fig. 1(b).

A more complex synthesis problem has been addressed in the successive example concerned with the design of an ADS with  $N = 90$ . The (90,8,0,59)-ADS in Fig. 2(a), whose autocorrelation is given in Fig. 2(b), has been obtained after  $i_{opt} = 108$  iterations as indicated by Fig. 2(c) where the iterative evolution of the optimal value of the fitness function,  $\Omega^i = \min_p \{F(\underline{b}_p^i)\}$ , is shown. For

completeness, let us consider that the whole GA-based minimization required the evaluation of about 9000 trial solutions to sample a search space of  $U = 1.23 \times 10^{27}$  sequences.

The last example is concerned with a higher-dimension problem, being  $N = 200$ . As expected, the number of iterations to identify  $\underline{b}_{opt}$  increases with respect to previous experiments [Fig. 3(a)] even though less than  $i_{opt} = 350$  are needed to fit the three-level autocorrelation function in Fig. 3(b). The (200,8,0,143)-ADS sequence is shown in Fig. 3(c).

*Conclusions:* In this letter, a GA-based technique has been proposed as a new methodological tool for designing ADS sequences of arbitrary length. The original synthesis has been reformulated as a combinatorial optimization. Towards this end, a suitable fitness function exploiting the autocorrelation properties of ADSs has been introduced and minimized by means of a GA-based iterative procedure. Selected numerical results have been reported to give a preliminary assessment of the capabilities and effectiveness of the proposed approach.

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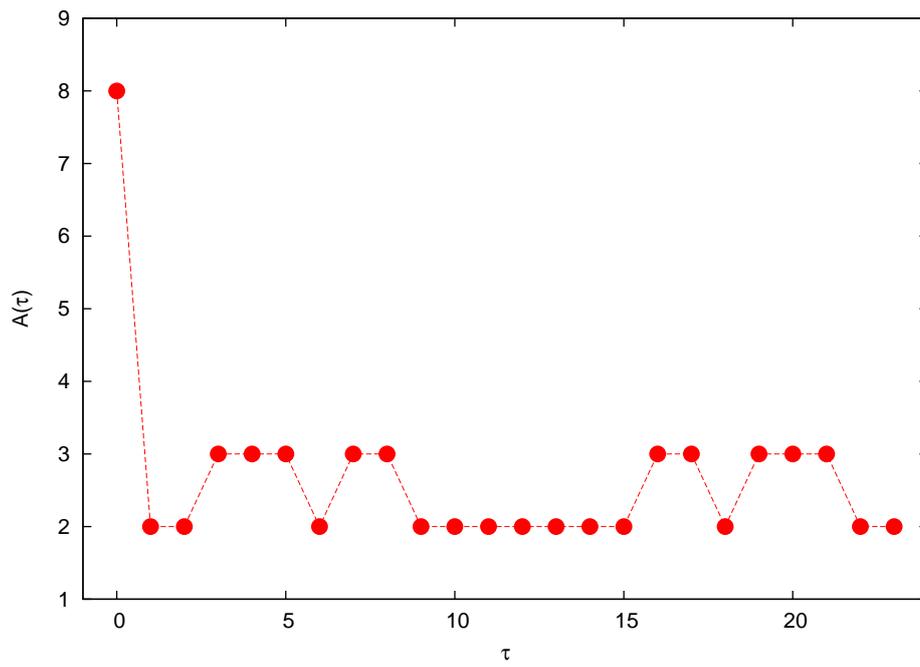
**Figure captions:**

Fig. 1 - *Numerical validation* ( $N = 24$ ) - (24,8,2,13)-ADS: (a) the autocorrelation function,  $A_{opt}(\tau)$ , and (b) the binary sequence,  $\underline{b}_{opt}$ .

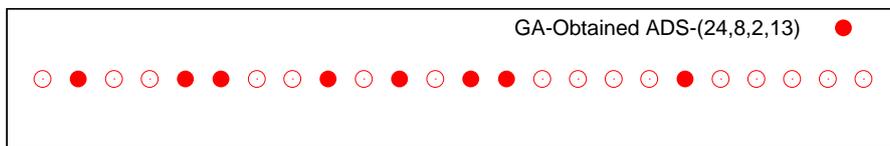
Fig. 2 - *Numerical validation* ( $N = 90$ ) - (90,8,0,59)-ADS: (a) the binary sequence,  $\underline{b}_{opt}$ , (b) the autocorrelation function,  $A_{opt}(\tau)$ , and (c) the optimal fitness,  $\Omega^i$ , versus the iteration index,  $i$ .

Fig. 3 - *Numerical validation* ( $N = 200$ ) - (200,8,0,143)-ADS: (a) the optimal fitness,  $\Omega^i$ , versus the iteration index,  $i$ , (b) the autocorrelation function,  $A_{opt}(\tau)$ , and (c) the binary sequence,  $\underline{b}_{opt}$ .

Figure 1

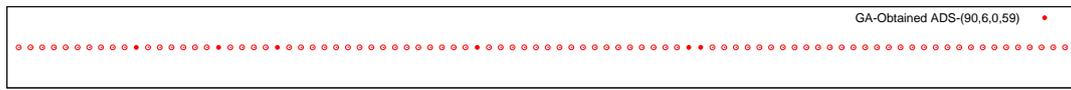


(a)

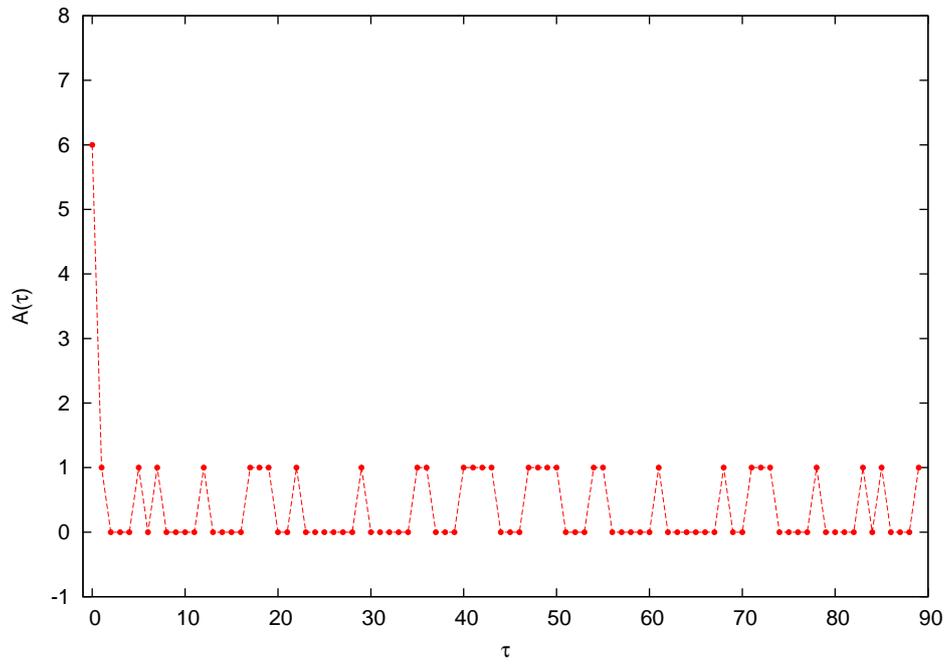


(b)

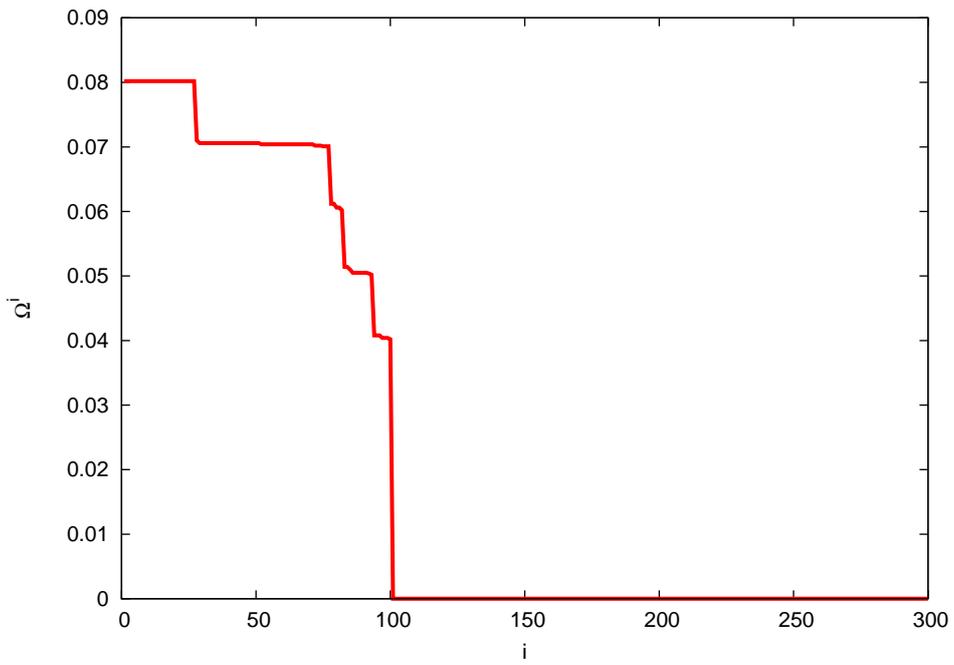
Figure 2



(a)

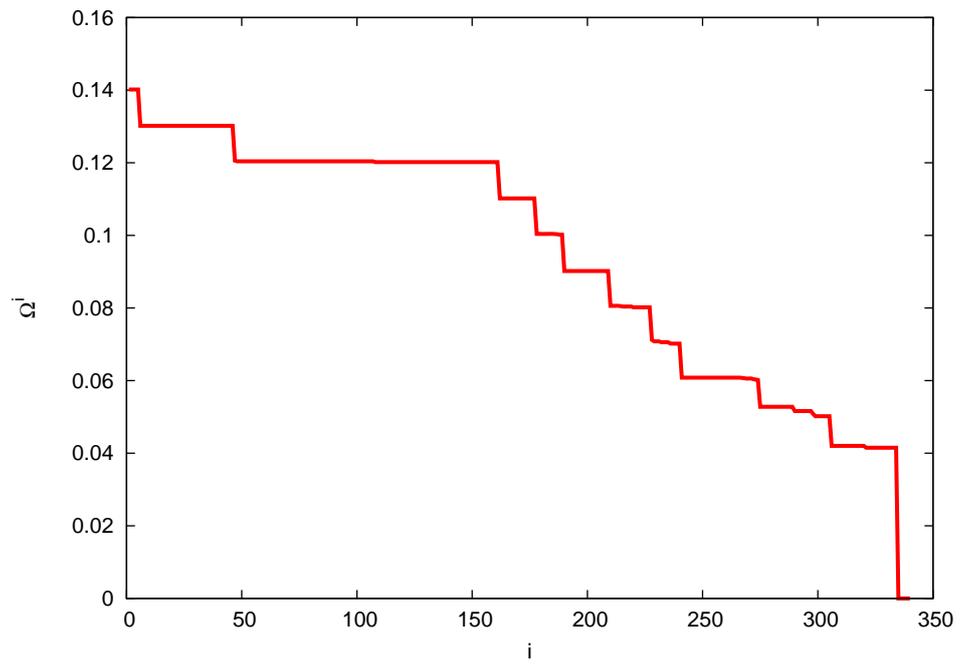


(b)

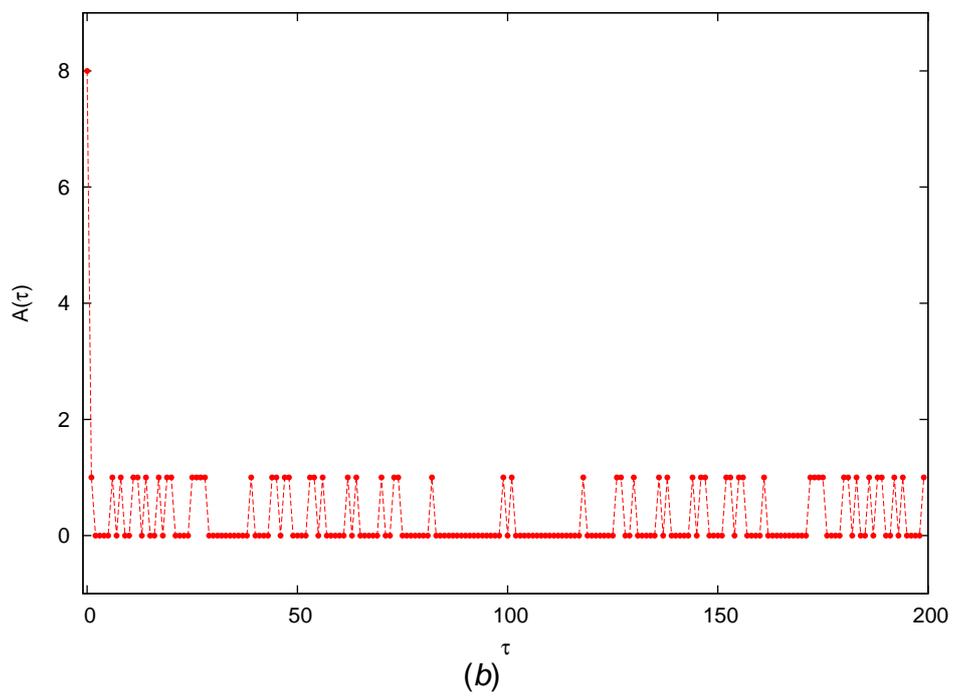


(c)

Figure 3



(a)



(b)



(c)