UNIVERSITY
OF TRENTO

DIPARTIMENTO DI INGEGNERIA E SCIENZA DELL'INFORMAZIONE

38123 Povo – Trento (Italy), Via Sommarive 14
http://www.disi.unitn.it

PERCOLATION-BASED MODELS FOR RAY-OPTICAL PROPAGATION IN STOCHASTIC DISTRIBUTIONS OF SCATTERERS WITH RANDOM SHAPE

A. Martini, L. Caramanica, M. Franceschetti, and A. Massa

January 2011

Technical Report # DISI-11-056

Percolation-Based Models for Ray-Optical Propagation in Stochastic Distributions of Scatterers with Random Shape

Anna Martini, Federico Caramanica, Massimo Franceschetti, and Andrea Massa,

Abstract

This letter deals with ray propagation in stochastic distributions of discrete scatterers having random shapes. The propagation medium is described by means of a semi-infinite percolating lattice and two different propagation models are considered. The propagation depth inside the medium is analytically estimated in terms of the probability that a ray reaches a prescribed level before being reflected back in the above empty half-plane. A comparison with Monte-Carlo-like experiments validate the proposed solutions. Applications are in wireless communications, remote sensing, and radar engineering.

Index Terms

Percolation theory, Stochastic ray tracing, Non-uniform random media, Scatterers with random shape.

I. Introduction

In the last years, several models based on the percolation theory [1] have been proposed to describe the electromagnetic wave propagation inside stochastic distributions of discrete scatterers more suitable to be stochastically modeled rather than being deterministically characterized [2]-[5]. In these works, the propagation medium is described by means of a random lattice of square sites (i.e., a grid whose cells may be occupied according to a known probability distribution) and the obstacles are assumed to be large with respect to the wavelength. Such an assumption allows to describe the electromagnetic wave radiated by the source as a collection of propagating rays that undergo specular reflections on the occupied sites.

The approach proposed in this paper follows the above description, but unlike [2]-[5] the rays are not reflected specularly and two different propagation models are presented. In the first model, referred as *Intrinsically-Square Shape Scatterers Model* (ISM), the ray is reflected back with a random angular direction, thus modeling propagation in a stochastic distribution of scatterers having sections with some random irregularities in an intrinsically-square shape. In the second model, indicated as *Completely Random Shape Scatterers Model* (CRM), whenever a ray hits an occupied cell, it enters the cell and then it escapes from a random point on the cell perimeter and with a random angular direction, thus describing propagation when obstacles are centered in a grid, but have completely random shapes.

The ISM and the CRM may be profitably used in several practical problems arising in wireless communications, remote sensing, and radar engineering provided that the dimensions of the obstacles are large enough to allow the optical approximation. As far as the first typology of problems is concerned, the ISM can model the propagation in a residential area whose buildings basically have the same orientation and square sections, but present some irregularities such as recesses and balconies, while the CRM can be adopted in describing the urban old town centers where each building has its own different orientation and shape. Concerning the applications in remote sensing and radar engineering, thanks to the recent access to the terahertz spectral range, the ensemble of practical problems that can be dealt with by relying on the proposed solutions is getting larger and

This work has been supported in Italy by the Progettazione di un Livello Fisico 'Intelligente' per Reti Mobili ad Elevata Riconfigurabilità, Progetto di Ricerca di Interesse Nazionale - MIUR Project COFIN 2005099984.

Authors are with the Department of Information and Communication Technology, University of Trento, Trento, Italy (e-mail: andrea.massa@ing.unitn.it, anna.martini@dit.unitn.it). M. Franceschetti is also with the Department of Electrical and Computer Engineering, University of California at San Diego, USA (e-mail: massimo@ece.ucsd.edu)

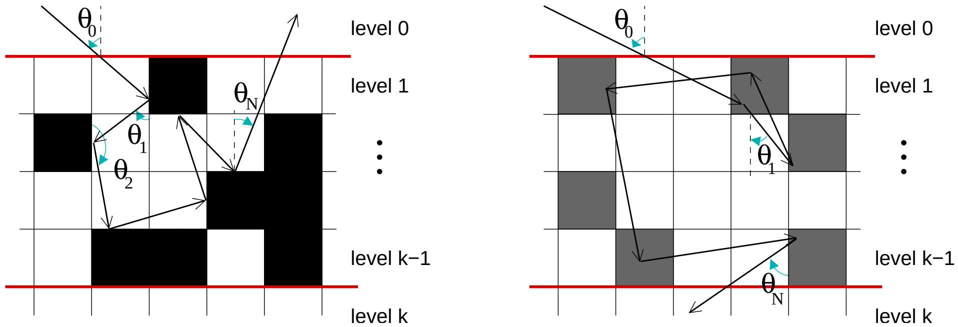

Fig. 1. Examples of ray propagation in random lattice realizations when the ISM (left) and the CRM (right) are assumed, respectively.

larger, since the wavelength goes down to the order of micrometers. In such a framework, of particular interest is the CRM that can model propagation in media such as rain, ice pellets, and granular soil.

II. Mathematical Formulation

Let us consider a semi-infinite percolation lattice of square sites that are occupied according to a known probability density distribution, $q_j = 1 - p_j$, j being the row index. A monochromatic plane wave, modeled as a collection of parallel rays, impinges on the lattice with incidence angle θ_0 (Fig. 1). The aim is to analytically estimate the probability that a single ray reaches a given level k inside the lattice before being reflected back into the above empty half-plane, $\Pr\{0 \mapsto k\}$.

In order to deal with the propagation models proposed in this letter, the so-called *Martingale approach* (i.e., the mathematical formulation proposed in [2] and generalized in [5] to the non-uniform case) is applied. The ray propagation inside the lattice is mathematically modeled in terms of the following stochastic process:

$$r_n = r_0 + \sum_{m=1}^{n} x_m, \qquad n \geq 0, \tag{1}$$

where r_n is the lattice row reached at the n-th reflection, r_0 is the row where the first reflection ($n = 0$) takes place, and $x_n = r_n - r_{n-1}$, $n \geq 1$, is a sequence describing the change of level between successive reflections. With reference to such a stochastic process, we can write

$$\Pr\{0 \mapsto k\} = \sum_{i=1}^{\infty} \Pr\{r_N \geq k \,|r_0 = i\} \Pr\{r_0 = i\}, \tag{2}$$

where N is defined as $N = \min\{n :\ r_n \geq k \text{ or } r_n \leq 0\}$, see Fig. 1. While the probability mass function $\Pr\{r_0 = i\}$ is exactly evaluated, the conditional probability $\Pr\{r_N \geq k\,|r_0 = i\}$ is estimated by applying the Martingale random processes theory [6] and the so-called Wald approximation as follows:

$$\Pr\{r_N \geq k\,|r_0 = i\} \cong \frac{\Delta_0}{\Delta_0 + \Delta_k}, \quad 0 < i < k, \tag{3}$$

Δ_0 and Δ_k being the distances that the ray, starting from level $r_0 = i$, needs to cover before escaping from the grid or reaching level k, respectively [5]. Such distances must be clearly meant as the number of obstacles that oppose the ray path towards level 0 and k, respectively.

In the following, the Martingale approach is applied to the ISM and to the CRM.

A. *Intrinsically-Square Shape Scatterers Model (ISM)*

Let us assume that at the $(n-1)$-th reflection, the ray is reflected back from the same incidence point, but with a random orientation θ_n. By following the convention graphically described in the left-hand side of Fig. 1, θ_n is uniformly distributed between -90^o and 90^o, when the reflection occurs on a horizontal face, and between 0^o and 180^o, when the reflection takes place on a vertical face.

As far as $\Pr\{r_0 = i\}$ is concerned, it is trivial to observe that, until a reflection occurs, the problem at hand comes down to the canonical one with specular reflections. Accordingly [5],

$$\Pr\{r_0 = i\} = p_1 q^+_{e_i,0} \prod_{j=1}^{i-1} p^+_{e_j,0}, \quad i \geq 1, \tag{4}$$

where $p^+_{e_j,0} = p_j^{|\tan\theta_0|} p_{j+1} = 1 - q^+_{e_j,0}$ is the effective probability that the ray crosses level j reaching level $j+1$ by proceeding in the positive direction with angle θ_0.

Now, let us estimate $\Pr\{r_N \geq k\,|r_0 = i\}$ according to (3). Since whenever a ray hits an horizontal face it is always reflected back changing its direction of propagation, all the horizontal faces must be counted as obstacles. As far as the vertical faces are concerned, let us focus on a single level. Whatever the number of reflections on vertical faces, the ray changes its direction of propagation with probability $1/2$, since θ_n is uniformly distributed between 0^o and 180^o for all $n \geq 1$, see Fig. 1. Therefore and taking into account that a ray traveling with a negative direction through level 1 surely escapes the grid, being the horizontal face between level 1 and level 0 surely empty, it follows that

$$\Delta_0 = \frac{i}{2} + (i-1) = \frac{3i-2}{2}, \tag{5}$$

$$\Delta_k = \frac{k-i}{2} + (k-i) = \frac{3(k-i)}{2}, \tag{6}$$

and thus,

$$\Pr\{r_N \geq k\,|r_0 = i\} \cong \frac{3i-2}{3k-2}. \tag{7}$$

By substituting (7) and (4) in (2), after simple manipulations as those reported in Appendix B of [4], we get

$$\Pr\{0 \mapsto k\} = p_1 \sum_{i=1}^{k-1} \frac{3i-2}{3k-2} q^+_{e_i,0} \prod_{j=1}^{i-1} p^+_{e_j,0} + p_1 \prod_{j=1}^{k-1} p^+_{e_j,0}, \tag{8}$$

which in the uniform case reduces to

$$\Pr\{0 \mapsto k\} = \frac{p\left[1 - 3p^k_{e,0} + 2p_{e,0}\right]}{q_{e,0}(3k-2)}, \tag{9}$$

with $p_{e,0} = p^{|\tan\theta_0|+1} = 1 - q_{e,0}$. According to the analysis provided in Appendix, (8) and (9) are expected to hold true with an increasing precision when: (*a*) $\theta_0 \to 90^o$ or $n \to \infty$, (*b*) the grid is dense, and (*c*) the probability density profile q_j, $j \geq 1$, does not present discontinuities and a significant variation throughout the lattice.

B. Completely Random Shape Scatterers Model (CRM)

With reference to the right-hand side of Fig. 1, let us assume that whenever a ray hits an occupied cell, it first enters the cell and then it escapes from a point on the cell perimeter and with an angle θ_n both modeled as uniformly distributed random variables.

In order to evaluate $\Pr\{r_0 = i\}$, we need to take into account the following difference with respect to the ISM (and to the canonical model [5] as well): the ray enters a cell also when an obstacle is present, see Fig. 1. Accordingly, since at each level the average number of cells on the ray-path is $|\tan\theta_0| + 1$, the effective probability that a ray freely crosses a level j reaching the following one without any reflection is equal to $p_{e_j,0} = p_j^{|\tan\theta_0|+1} = 1 - q_{e_j,0}$. It easily follows that

$$\Pr\{r_0 = i\} = q_{e_i,0} \prod_{j=1}^{i-1} p_{e_j,0}, \quad i \geq 1. \tag{10}$$

Now, let us consider $\Pr\{r_N \geq k\,|r_0 = i\}$. In this case obstacles must be considered as cells and the cells number at each

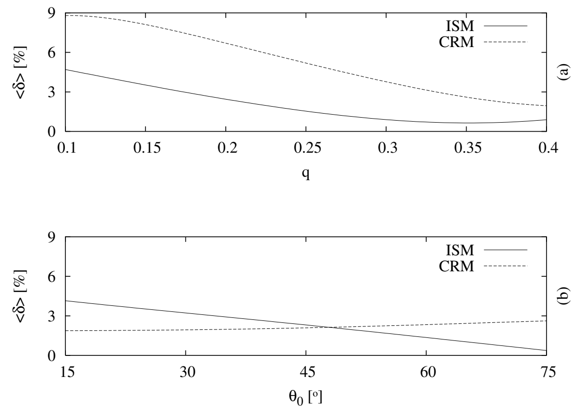

Fig. 2. Uniform case - Mean error $\langle\delta\rangle$ versus q when $\theta_0 = 75^o$ (a) and mean error $\langle\delta\rangle$ versus θ_0 when $q = 0.35$ (b).

level is the same throughout the whole lattice. Thus, (3) takes the form

$$\Pr\left\{r_N \geq k \left| r_0 = i \right.\right\} = \frac{i}{k}, \tag{11}$$

and after simple manipulations as those reported in Appendix B of [4], we obtain

$$\Pr\{0 \mapsto k\} = \sum_{i=1}^{k-1} \frac{i}{k} q_{e_i,0} \prod_{j=1}^{i-1} p_{e_j,0} + \prod_{j=1}^{k-1} p_{e_j,0}, \tag{12}$$

which in the uniform case becomes

$$\Pr\{0 \mapsto k\} = \frac{1 - p_{e,0}^k}{q_{e,0} k}. \tag{13}$$

As detailed in the Appendix, (12) and (13) are expected to efficiently perform whatever the incidence angle θ_0 provided that (*a*) the grid is dense and (*b*) the probability density profile q_j, $j \geq 1$, does not present discontinuities and a significant variation throughout the lattice.

III. Numerical Validation

In order to assess the effectiveness of the proposed solutions, as well as their range of validity, an exhaustive set of experiments has been performed and selected representative results are reported in the following. As a reference, the propagation depth has been numerically estimated in the first $K = 32$ levels of the lattice and reliability of the analytical solutions described in the previous section has been quantitatively evaluated through the mean error $\langle\delta\rangle$

$$\langle\delta\rangle \triangleq \frac{1}{K} \sum_{k=1}^{K} \frac{\left|\Pr_N\left\{0 \mapsto k\right\} - \Pr_A\left\{0 \mapsto k\right\}\right|}{\max_k \left[\Pr_N\left\{0 \mapsto k\right\}\right]} \times 100, \tag{14}$$

where the sub-scripts N and A indicate numerically and analytically estimated values, respectively.

The first test case is aimed at analyzing how the obstacles density affects the performances. Towards this end, we considered uniform random grids with q varying from 0.1 up to 0.4[1] with step 0.05 and we fixed the incidence angle θ_0 to 75^o. With reference to Fig. 2(*a*), it can be observed that in both the cases and as expected [conditions (*b*), Sec. II.*A*, and (*a*), Sec. II.*B*], the reliability of the analytical formulation increases as q increases, the mean error going down to $\langle\delta\rangle_{ISM} = 0.37\%$ and $\langle\delta\rangle_{CRM} = 1.95\%$ when $q = 0.35$ and $q = 0.4$, respectively. With reference to the ISM model, it is worth noting that $\langle\delta\rangle_{q=0.4} > \langle\delta\rangle_{q=0.35}$, as in the canonical case of specular reflections [5].

The second test case is devoted to analyze the impact of the incidence angle. We considered uniform percolation lattices with q fixed to $q = 0.35$ and different incidence conditions, namely $\theta_0 = \{15^o, 30^o, 45^o, 60^o, 75^o\}$. The obtained values of $\langle\delta\rangle$

[1] No higher q values have been considered since for $p < p_c$, $p_c \approx 0.59275$ in the two-dimensional case, propagation is inhibited according to the phase transition property of percolation lattices [1].

TABLE I
LINEAR CASE - α VALUES AND $\langle\delta\rangle$ VALUES WHEN $\theta_0 = 75^o$.

Profile	L1	L2	L3
α [$\times 10^{-3}$]	3.23	6.46	9.68
$\langle\delta\rangle_{ISM}$ [%]	0.59	0.28	0.39
$\langle\delta\rangle_{CRM}$ [%]	3.18	3.75	4.50

are plotted in Fig. 2(*b*). As expected [condition (*a*), Sec. II.*A*], in the ISM case the performances are affected by the value of θ_0 ($\max_{\theta_0} \langle\delta\rangle / \min_{\theta_0} \langle\delta\rangle \cong 11.2$) and the mean error decreases as θ_0 increases. On the other hand, the results obtained for the CRM are insensitive to θ_0, being $\max_{\theta_0} \langle\delta\rangle / \min_{\theta_0} \langle\delta\rangle \cong 1.4$.

Finally, we analyze how performances are affected by the slope in the density profile. Towards this end, we considered three decreasing linear profiles $q_j = q - \alpha(j-1)$, q being equal to 0.4, having different angular coefficients α, see Tab. I. In particular, the α values were chosen so that the occupation probability of the last level, q_K, is equal to 0.3 (profile L1), 0.2 (profile L2), and 0.1 (profile L3). The incidence angle was fixed to $\theta_0 = 75^o$. The results in terms of $\langle\delta\rangle$ are given in Tab. I. Surprisingly [conditions (*c*), Sect. II.*A*, and (*b*), Sect. II.*B*] and unlike the canonical case [5], it turns out that the predictions accuracy is not sensitive to the slope in the density profile, but it depends only on the obstacles density throughout the lattice. In the ISM case, $\langle\delta\rangle$ does not change significantly with α, with values comparable to the best $\langle\delta\rangle$ values of the uniform case. On the other hand, in the CRM case, performances get worse as α increases, but this is only due to the fact that the density profile takes values farther and farther from the optimal value (i.e., $q_{opt} = 0.4$). In fact, the $\langle\delta\rangle$ values are comparable with those related to the uniform case ($\langle\delta\rangle_{Li,\, i=1,2,3} \cong \frac{\langle\delta\rangle_{q_1} + \langle\delta\rangle_{q_K}}{2}$, $\langle\delta\rangle_{q_x}$ being the mean error in correspondence with a uniform grid having $q = q_x$).

IV. CONCLUSION

In this letter, ray propagation through a stochastic non-uniform distribution of discrete scatterers with random shape has been considered. The environment has been described in terms of a percolation lattice and two different propagation models have been proposed. In both cases, an analytical closed-form solution for the penetration depth has been provided and assessed through numerical experiments. Summarizing, we can state that in the ISM case the proposed solution satisfactorily performs for dense grids and better and better as $\theta_0 \to 90^o$ or $n \to \infty$, while in the CRM case accuracy increases as q increases whatever the incidence angle.

APPENDIX

This Appendix is devoted at analyzing the range of validity of the proposed solutions. We start by recalling that (3), and thus the final results (8) and (12), hold true provided that the ray-jumps following the first one are independent, identically-distributed, and with mean and standard deviation approaching zero [5]. To understand when such an assumption is verified with good approximation, we need to analyze the probability mass function of the ray-jumps following the first one, $\Pr\{x_n = i\}$. In the following, we refer to the probabilities that the n-th jump is in positive or negative direction as $\Pr\{x_n^+\}$ and $\Pr\{x_n^-\}$, respectively. Moreover, we assume that x_n starts at level j, where the $(n-1)$-th reflection takes place, and consequently ends at level $j+i$, where the n-th reflection occurs.

Let us first analyze $\Pr\{x_n = i\}$ when the ISM is considered. According to the adopted notation, we can write

$$\Pr\{x_n = i\} = \begin{cases} \Pr\{x_n^+\}\, q^+_{e_j,n} + \Pr\{x_n^-\}\, q^-_{e_j,n}, & i = 0, \\ \Pr\{x_n^+\}\, q^+_{e_{j+i},n} \prod_{s=0}^{i-1} p^+_{e_{j+s},n}, & i > 0, \\ \Pr\{x_n^-\}\, q^-_{e_{j+i},n} \prod_{s=0}^{|i|-1} p^-_{e_{j-s},n}, & i < 0, \end{cases} \quad (15)$$

where $p^{-}_{e_j,n} = p_j^{|\tan\theta_n|} p_{j-1} = 1 - q^{-}_{e_j,n}$ is the effective probability of the level j to be freely crossed, given that the ray travels in the negative direction with angle θ_n.

At this point, we prove the following **Lemma**. *If* $\theta_0 \to 90^o$ *or* $n \to \infty$ *and* $p_j \cong p_{j+1}$, $\forall j$, *then* $\Pr\{x_n^+\} \cong \Pr\{x_n^-\} \cong 1/2$. We first observe that whenever a ray hits a horizontal face, it surely changes its direction of propagation. On the other hand, if the reflection occurs on a vertical face, the direction of propagation is either kept or changed with the same probability. Thus,

$$\begin{array}{c} \Pr\{x_n^+\} = \Pr\left\{x_n^+ \left| x_{n-1}^+ \right.\right\} \Pr\left\{x_{n-1}^+\right\} \\ + \Pr\left\{x_n^+ \left| x_{n-1}^- \right.\right\} \Pr\left\{x_{n-1}^-\right\} = \frac{1}{2}\xi_{v,n-1} \Pr\left\{x_{n-1}^+\right\} \\ + \left(\frac{1}{2}\xi_{v,n-1} + \xi_{h,n-1}\right) \Pr\left\{x_{n-1}^-\right\} \\ = \left(\frac{1}{2}\xi_{v,n-1} + \xi_{h,n-1}\right) - \xi_{h,n-1} \Pr\left\{x_{n-1}^+\right\}, \end{array} \tag{16}$$

where $\xi_{v,n}$ and $\xi_{h,n}$ denote the probabilities that the n-th reflection takes place on a horizontal and on a vertical face, respectively. Now, under the assumption that $p_j \cong p_{j+1}$, $\forall j$, and taking into account that the quantity $|\tan\theta_n|$ is a random variable identically distributed for all $n \geq 1$, we have

$$\xi_{v,n} \cong \frac{|\tan\theta_n|}{|\tan\theta_n| + 1} = \xi_v = 1 - \xi_h, \ \forall n, \, n \geq 1, \tag{17}$$

Accordingly, since $\Pr\left\{x_1^+\right\} = \frac{1}{2}\xi_{v,0} \cong \frac{1}{2}\frac{|\tan\theta_0|}{1+|\tan\theta_0|}$, being the first jump r_0 in positive direction, from (16) it follows that

$$\begin{array}{c} \Pr\{x_n^+\} = \left(\frac{1}{2}\xi_v + \xi_h\right) \displaystyle\sum_{i=0}^{n-2} (-\xi_h)^i + \Pr\left\{x_1^+\right\}(-\xi_h)^{n-1} \\ = \frac{1}{2}\left[1 - (-\xi_h)^{n-1}\right] + \frac{1}{2}\frac{|\tan\theta_0|}{1+|\tan\theta_0|}(-\xi_h)^{n-1}. \end{array} \tag{18}$$

At this point, it easily follows that $\Pr\{x_n^+\}$ tends to $1/2$ if either $n \to \infty$ or $\theta_0 \to 90^o$.

Now, let us observe (15) taking into account the **Lemma**. If $\theta_0 \to 90^o$ or $n \to \infty$ and $p_j \cong p_{j+1}$, $\forall j$, and if the additional condition $q_j \cong q_i$ holds true whatever i and j, in first approximation the assumption of independent, identically distributed and zero-mean jumps is satisfied. As far as the condition on the standard deviation is concerned, it is trivial to observe that it decreases as the obstacles density throughout the whole lattice increases.

The analysis of $\Pr\{x_n = i\}$ in the CRM case is now in order. Let us focus on the $(n-1)$-th reflection: the ray enters the cell and then it is reflected from any of the four cell sides with the same probability. Let N, E, S, and W be the events that the ray exits from the north, the east, the south, or the west side, respectively. It can be observed that if event N takes place, then $\Pr\{x_n \leq -1\} = 1$, while if event S occurs, then $\Pr\{x_n \geq 1\} = 1$. On the other hand, when E or W takes place, since θ_n is uniformly distributed between 0^o and 180^o for all $n \geq 1$, then $\Pr\{x_n^+\} = \Pr\{x_n^-\} = 1/2$. Therefore,

$$\Pr\{x_n = i\} = \begin{cases} \frac{1}{4}q_{e_j,n}, & i = 0, \\ \frac{1}{4}q_{e_{j+i},n}\displaystyle\prod_{s=1}^{i-1} p_{e_{j+s},n} + \frac{1}{4}q_{e_{j+i},n}\prod_{s=0}^{i-1} p_{e_{j+s},n}, & i > 0, \\ \frac{1}{4}q_{e_{j+i},n}\displaystyle\prod_{s=1}^{|i|-1} p_{e_{j-s},n} + \frac{1}{4}\prod_{s=0}^{|i|-1} p_{e_{j-s},n}, & i < 0, \end{cases} \tag{19}$$

Now, if the condition $q_j \cong q_i$ holds true whatever i and j, we can conclude that in first approximation the assumption of independent, identically distributed and zero-mean jumps is satisfied. Moreover, as in the ISM case, the standard deviation decreases as the obstacles density increases.

References

[1] G. Grimmett, *Percolation*. Springer-Verlag, New York, 1989.

[2] G. Franceschetti, S. Marano, and F. Palmieri, "Propagation without wave equation toward an urban area model," *IEEE Trans. Antennas Propag.*, vol. 47, pp. 1393-1404, 1999.

[3] S. Marano and M. Franceschetti, "Ray propagation in a random lattice: a maximum entropy, anomalous diffusion process," *IEEE Trans. Antennas Propag.*, vol. 53, pp. 1888-1896, 2005.

[4] A. Martini, M. Franceschetti, and A. Massa, "Ray propagation in nonuniform random lattices," *J. Opt. Soc. Amer. A*, vol. 23, pp. 2251-2261, 2006.

[5] A. Martini, R. Azaro, M. Franceschetti, and A. Massa, "Ray propagation in nonuniform random lattices. Part II," *J. Opt. Soc. Amer. A*, vol. 24, pp. 2363-2371, 2007.

[6] R. M. Ross, *Stochastic Processes.* J. Wiley, New York, 1983.