Failure Correction in Linear Arrays based on Compressive Processing

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1 Numerical Results - Non-Iterative MCFC

1.1 Test case 6—Comparison with [Rodriguez 2000], N=50, 3 faulty elements

1.1.1 Goal of the analysis

The goal of test case 6 is that of comparing the MFC method developed using the technique presented by Rodriguez *et al.* in [Rodriguez.2000] as reference. The said technique is based on Genetic Algorithms, and also tries to minimize the number of elements changed. Therefore, it is reasonable to expect that the solutions of the two methods are similar. However, it must be noted that the reference method uses a cost function also depending on the directivity and the maximum element-to-element excitation ratio.

1.1.2 Parameters

The array considered in test case 6 has the following properties

- Number of array elements: $N = 50$
- Tapering: Dolph-Chebyshev, SLL=-25 [dB]
- Damaged element indexes set: $\Omega = \{8, 18, 38\}$
- Number of faulty elements: $D = 3$
- Damaged element excitation: $\mathbf{w}_{\text{corr,immut}} = [0, 0, 0]$

Figure 1 shows the original excitations and the damaged ones.

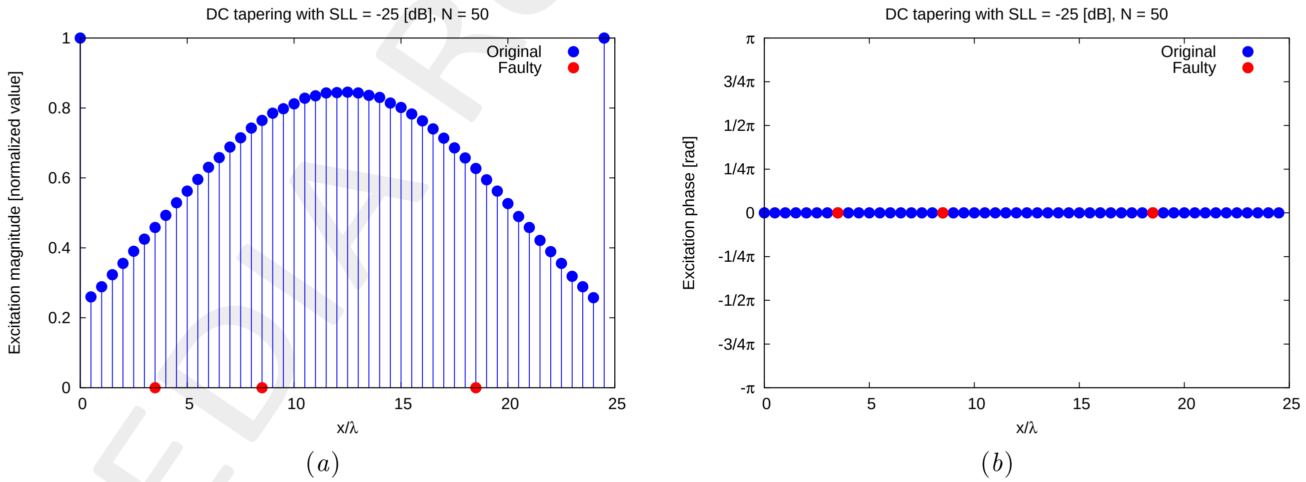


Figure 1: Original and damaged excitations for the array considered in test case 6: amplitude (a) and phase (b).

The parameters used to configure the software are the following:

- Phase 1

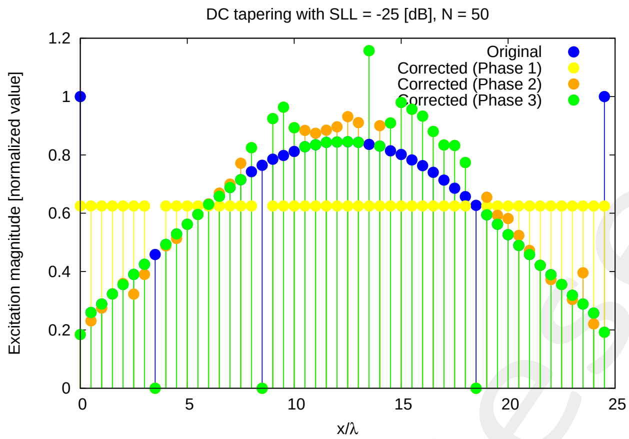
- Desired SLL: $\text{SLL}^{(1)} = -25.0$ [dB]
- Mask main lobe width: $\text{BW}^{(1)} = 6.7$ [deg]
- Mask u samples count: $K^{(1)} = 500$

- Phase 2
  - Desired SLL: $\text{SLL}^{(2)} = -24.5$ [dB]
  - Mask main lobe width: $\text{BW}^{(2)} = 6.7$ [deg]
  - Mask u samples count: $K^{(2)} = 500$

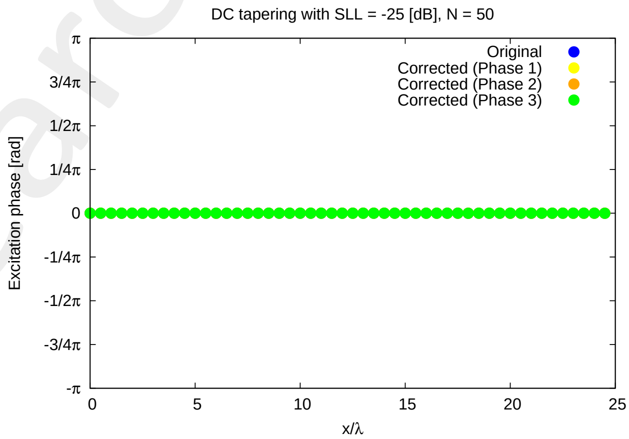
- Use Hessian: Yes

1.1.3 Results

Figure 2 compares the original excitations with the corrected excitations obtained with the proposed method.



(a)



(b)

Figure 2: Original and corrected excitations for the array considered in test case 1: amplitude (a) and phase (b).

Figure 3 compares the original, faulty and corrected radiation patterns.

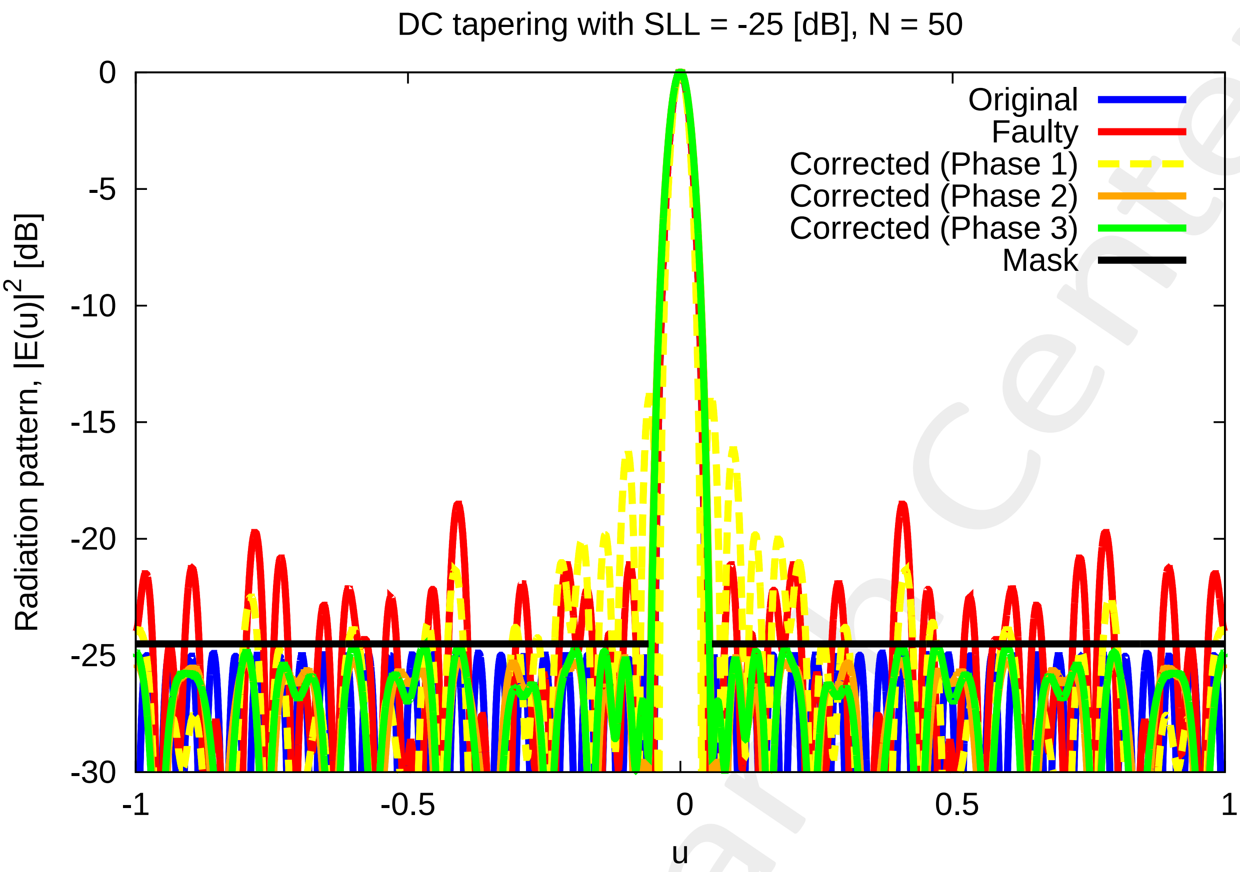


Figure 3: The radiation pattern for the original, faulty and corrected excitations.

Figure 4 shows the value of the L1-norm cost function for each iteration of the algorithm.

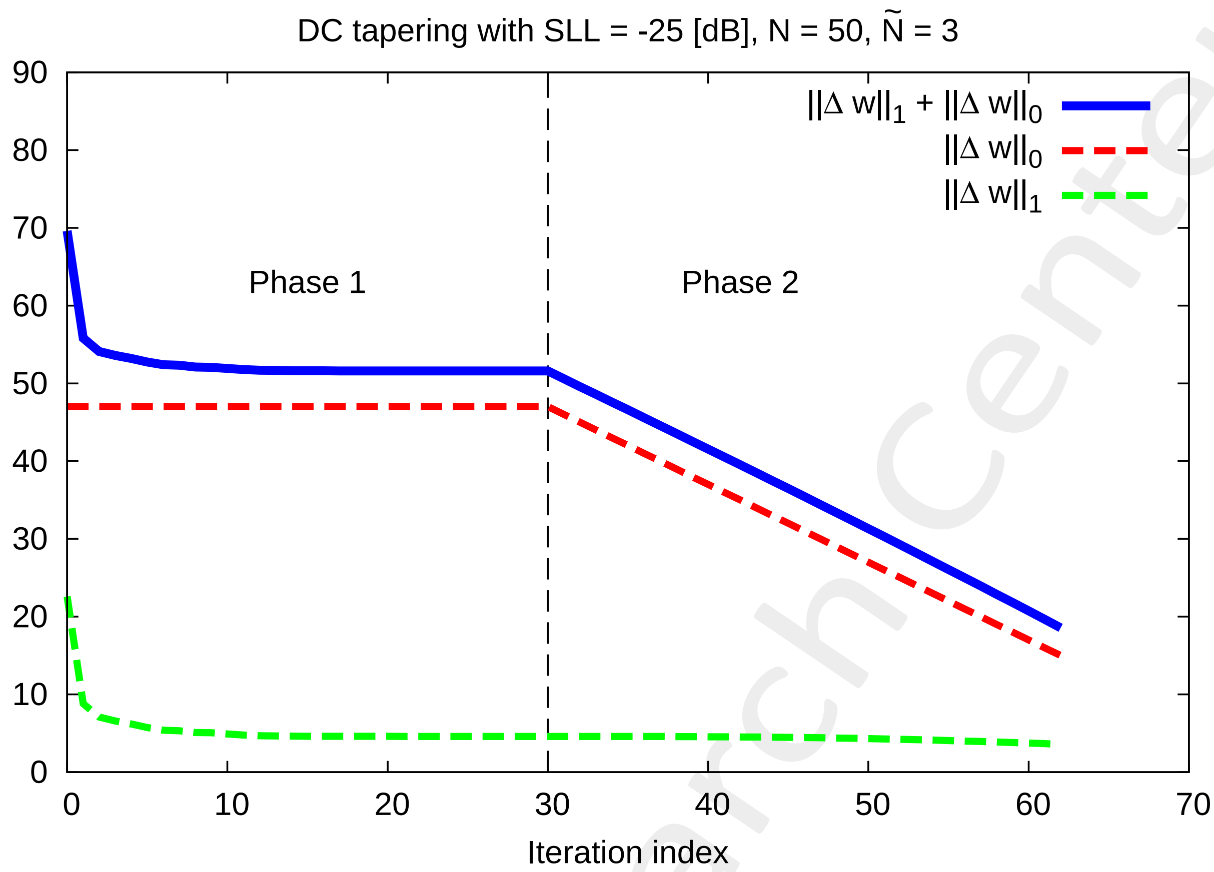


Figure 4: The value of the L1-norm cost function for each iteration of the algorithm.

Table I reports the SLL of the radiation patterns for the original, faulty and corrected excitations.

	Pattern SLL [dB]	**HPBW [deg]**	**DRR**	$\left\|\mathbf{w}_{\text{corr,mut}} - \mathbf{w}_{\text{orig,mut}}\right\|_1$	$\left\|\mathbf{w}_{\text{corr}}$
Original excitations	-24.90	2.28	0.258		
Faulty excitations	-18.51	2.28	0.258		
Corrected excitations (Step 1)	-13.89	2.02	1.0	8.48	
Corrected excitations (Phase 2)	-25.10	2.56	0.159	4.59	
Corrected excitations (Step 3)	-24.58	2.53	0.159	3.55	
State of the art [Rodriguez.2000]	-24.47	2.49	0.198	2.67	

Table I: Comparison of the original, faulty and corrected excitations.

1.1.4 Observations

The proposed method succeeded in providing a set of corrected excitations that matches the SLL and pattern peak of the original set of excitations. The SLL the solution obtained matches that of the reference solution, reported in [Rodriguez.2000]. However, the number of excitations changed is higher (33 vs 12).

1.2 Test case 7—Comparison with [Rodriguez 2000], N=50, 3 faulty elements, max 5 corrections

1.2.1 Goal of the analysis

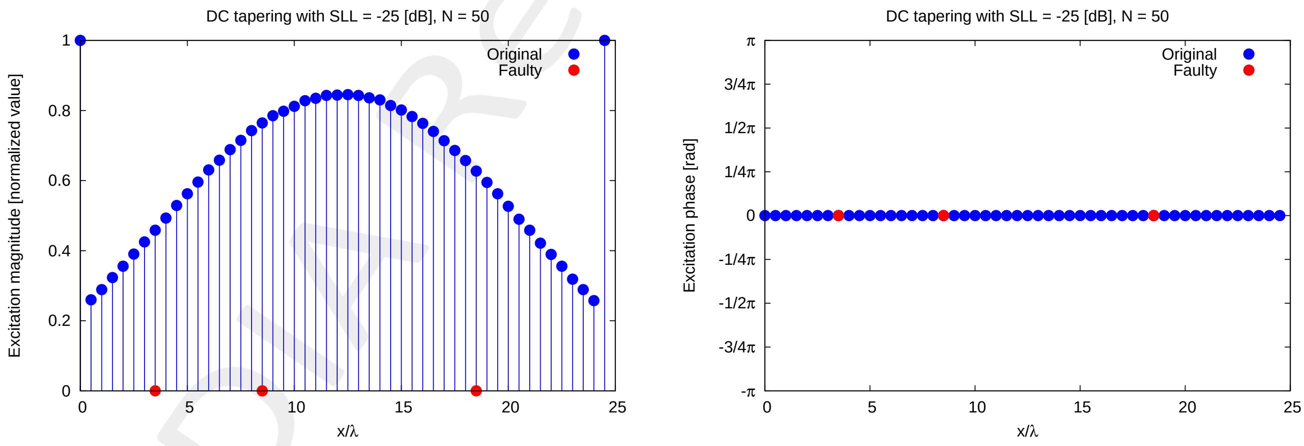
The goal of test case 7 is that of comparing the MFC method developed using the technique presented by Rodriguez *et al.* in [Rodriguez.2000] as reference. The said technique is based on Genetic Algorithms, and also tries to minimize the number of elements changed. Therefore, it is reasonable to expect that the solutions of the two methods are similar. However, it must be noted that the reference method uses a cost function also depending on the directivity and the maximum element-to-element excitation ratio.

1.2.2 Parameters

The array considered in test case 7 has the following properties

- Number of array elements: $N = 50$
- Tapering: Dolph-Chebyshev, SLL$=-25$ [dB]
- Damaged element indexes set: $\Omega = \{8, 18, 38\}$
- Number of faulty elements: $D = 3$
- Damaged element excitation: $\mathbf{w}_{\text{corr,immut}} = [0, 0, 0]$

Figure 5 shows the original excitations and the damaged ones.



(*a*) (*b*)

Figure 5: Original and damaged excitations for the array considered in test case 7: amplitude (*a*) and phase (*b*).

The parameters used to configure the software are the following:

- Phase 1
  - Desired SLL: SLL$^{(1)} = -24$ [dB]

– Mask main lobe width: $BW^{(1)} = 6.7$ [deg]

– Mask u samples count: $K^{(1)} = 500$

- Phase 2

  – Desired SLL: $SLL^{(1)} = -22.4$ [dB]

  – Mask main lobe width: $BW^{(1)} = 6.7$ [deg]

  – Mask u samples count: $K^{(1)} = 500$

- Use Hessian: Yes

1.2.3 Results

Figure 6 compares the original excitations with the corrected excitations obtained with the proposed method.

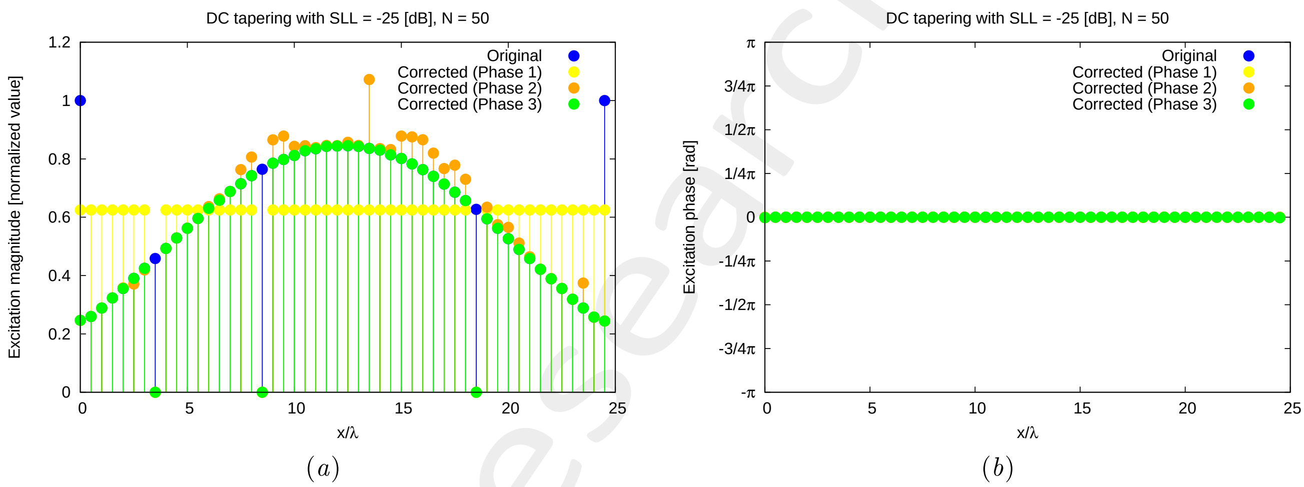


Figure 6: Original and corrected excitations for the array considered in test case 1: amplitude (a) and phase (b).

Figure 7 compares the original, faulty and corrected radiation patterns.

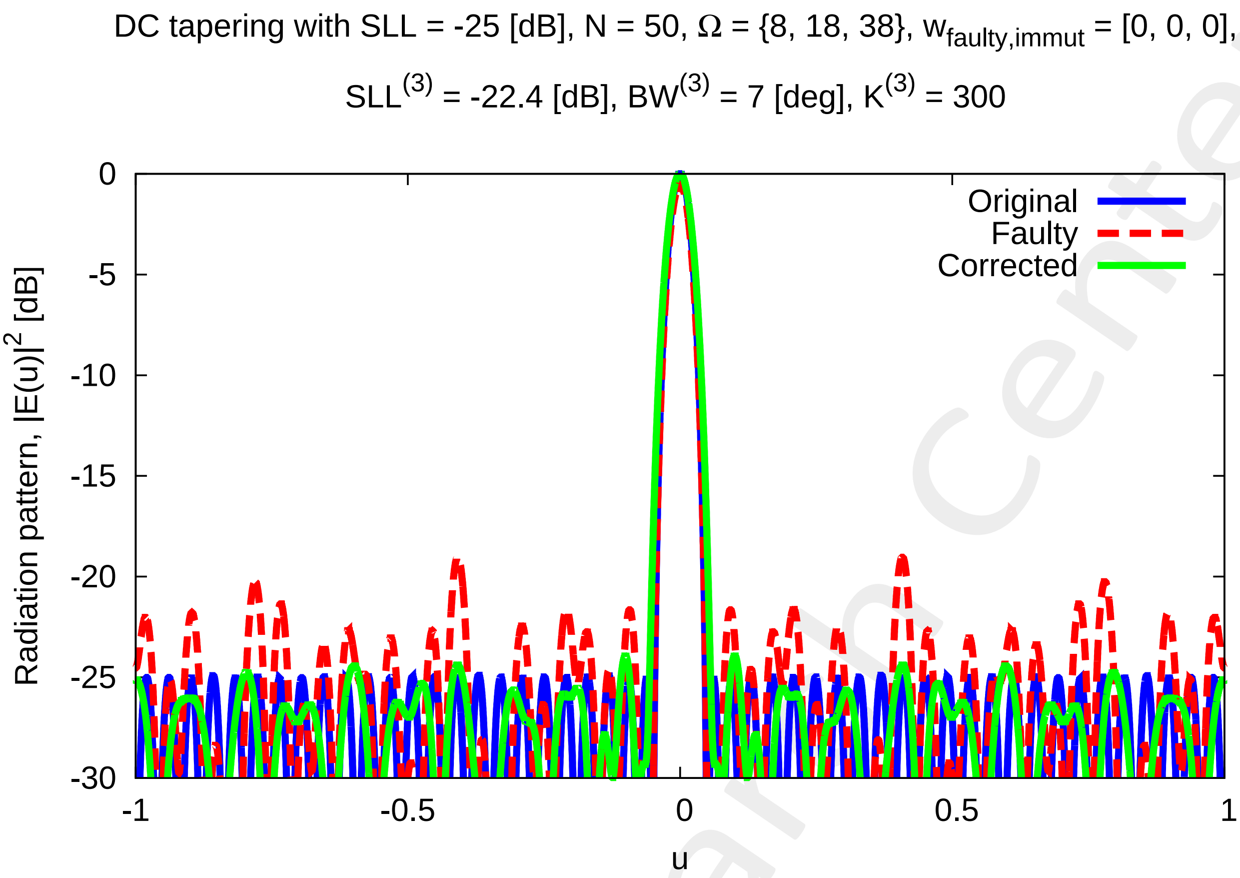


Figure 7: The radiation pattern for the original, faulty and corrected excitations.

Figure 8 shows the value of the L1-norm cost function for each iteration of the algorithm.

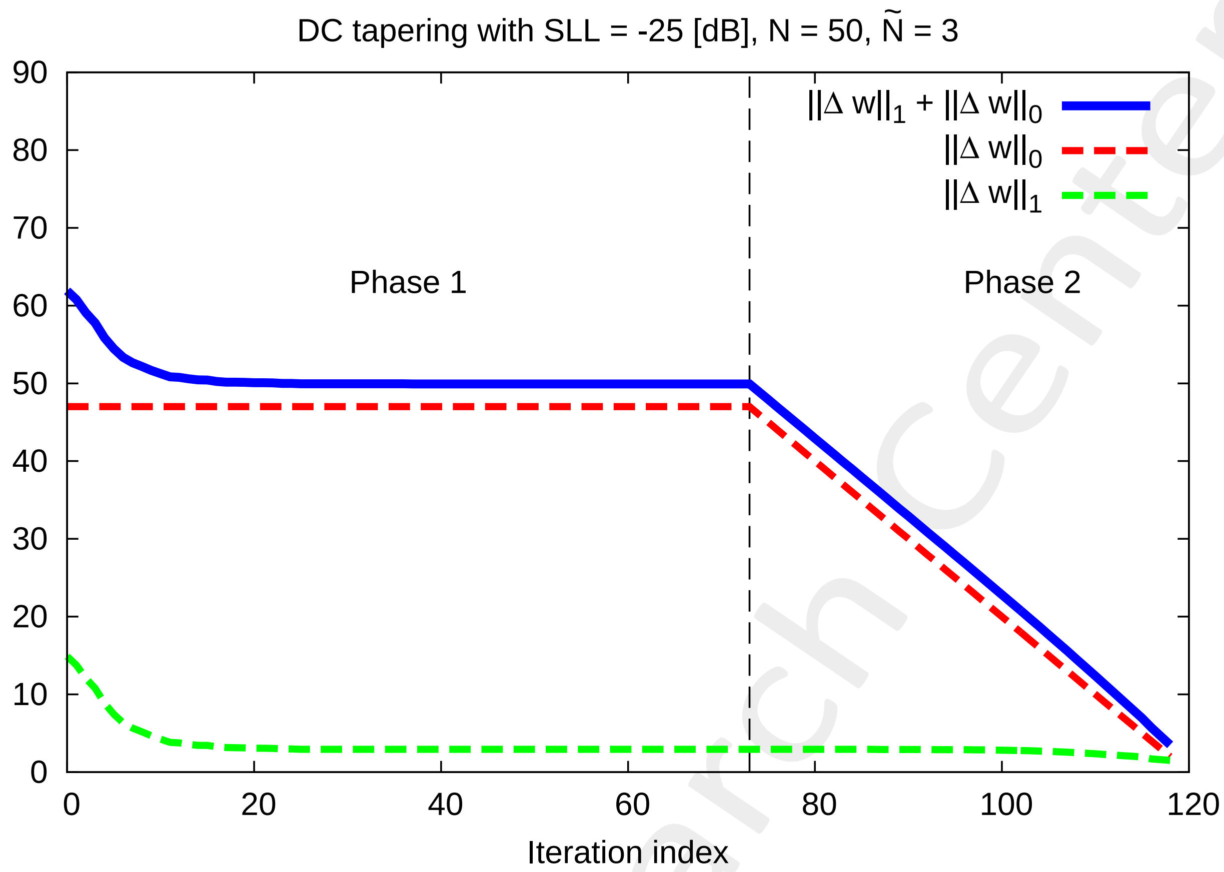


Figure 8: The value of the L1-norm cost function for each iteration of the algorithm.

Table II reports the SLL of the radiation patterns for the original, faulty and corrected excitations.

	Pattern SLL [dB]	**HPBW [deg]**	**DRR**	$\left\|\mathbf{w}_{corr,mut} - \mathbf{w}_{orig,mut}\right\|_1$	$\left\|\mathbf{w}_{corr}$
Original excitations	-24.90	2.28	0.258		
Faulty excitations	-18.51	2.28	0.258		
Corrected excitations (init.)	-13.89	2.02	1.0	8.48	
Corrected excitations (Phase 1)	-24.36	2.48	0.228	2.93	
Corrected excitations (Phase 2)	-22.77	2.46	0.289	1.51	
State of the art [Rodriguez.2000]	-22.4	*n.a.*	*n.a.*	*n.a.*	

Table II: Comparison of the original, faulty and corrected excitations.

1.2.4 Observations

The proposed method succeeded in providing a set of corrected excitations. Moreover, it provided a solution whose SLL matches that of the reference solution, reported in [Rodriguez.2000], when the number of corrected elements is constrained to be less than 5. Finally, only 2 excitation corrections are required by the proposed method.

2 Conclusions

The failure correction problem was formalized and a novel failure correction technique was introduced. The technique aims at producing a corrected pattern which satisfies certain SLL and main lobe beamwidth requirements while minimizing the number of corrected elements.

In Section 1, a number of test cases were reported to study the strengths and weaknesses of the presented technique, along with brief discussions of our findings. The proposed method was compared to other state-of-the-art method found in the literature. Furthermore, the presented technique was tested against arrays of different size, showing that an increase in array elements results in a reduction in the relative number of corrected excitations.

Future work:

- The l_1-minimization is convex, but in this application we have non-linear constraints. Is there a way to reformulate the SLL constraints so that they are at least quadratic? Or maybe use a different metric which is quadratic? If that is possible, the problem belongs to the class of quadratically constrained quadratic programming (QCQP), where, under some conditions, globallly optimal solutions can be found.

- A branch-and-cut algorithm in place of the backtracking algorithm for the selection of the elements to correct could be theoretically proved to be optimal. In particular, assume we have 4 elements that can be corrected ($N_C = 4$), but we don't the minimum number of elements that need corrections, $\widehat{N}_C$, nor which elements to corrected. There are

$$\sum_{\widehat{N_C}=1}^{N_C} \binom{N_C}{\widehat{N}_C} = \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 15 \tag{1}$$

  possible combinations of corrections that need to be tested. However, if we verify that the set of corrections $\{1, 2, 3\}$ does not work, than we can exclude that all corrections sets that are a subset of $\{1, 2, 3\}$ will not work either (i.e. $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$). This means that a whole "branch" of solutions can be excluded or "cut", reducing the solution space. Careful choice of which combinations to test might substantially reduce the number of candidate solutions that require testing. Furthermore, assuming the algorithm satisfying the constraints is optimal, then also the branch-and-cut algorithm for the choice of the corrections becomes optimal.

- At the present, the MCFC method allows different metric for the constraints, but only a combination of SLL/BW was tested. Future work might consider the Directivity as or the Link Capacity as an alternative metric.

- At the present, the MCFC method considers isotropic radiators. Future work might extend the method to consider real elements.

3 Appendix

3.1 1st and 2nd order derivatives of the L1 cost function

3.1.1 L1 cost function as a function of real variables

The cost function for the MFC problem is defined as

$$f(\mathbf{w}_{\text{corr,mut}}) \triangleq \left\| \mathbf{w}_{\text{corr,mut}} - \mathbf{w}_{\text{orig,mut}} \right\|_1 = \sum_{n=1}^{N-D} \left| w_{\text{corr,mut},n} - w_{\text{orig,mut},n} \right| \tag{2}$$

Since the Matlab function `fmincon()` only works on real variables, the cost function will be rewritten as a function of the real and imaginary components of $\mathbf{w}_{\text{corr,mut}}$,

$$\mathbf{a} \triangleq \text{Re}\{\mathbf{w}_{\text{corr,mut}}\} \in \mathbb{R}^{\mathbb{N}-\mathbb{D}} \quad \text{and} \quad \mathbf{b} \triangleq \text{Im}\{\mathbf{w}_{\text{corr,mut}}\} \in \mathbb{R}^{\mathbb{N}-\mathbb{D}} \tag{3}$$

The cost function will thus be

$$\begin{aligned} g(a_1, \ldots, a_{N-D}, b_1, \ldots, b_{N-D}) = g(\mathbf{a}, \mathbf{b}) = \quad & \triangleq \sum_{n=1}^{N-D} \left| a_n + jb_n - w_{\text{orig,mut},n} \right| \\ & = \sum_{n=1}^{N-D} \sqrt{\text{Re}\left\{ a_n + jb_n - w_{\text{orig,mut},n} \right\}^2 + \text{Im}\left\{ a_n + jb_n - w_{\text{orig,mut},n} \right\}^2} \\ & = \sum_{n=1}^{N-D} \sqrt{\left(a_i - \text{Re}\left\{ w_{\text{orig,mut},i} \right\} \right)^2 + \left(b_i - \text{Im}\left\{ w_{\text{orig,mut},i} \right\} \right)^2} \end{aligned}$$

3.1.2 Gradient of the L1 cost function

The gradient of $g(\mathbf{a}, \mathbf{b})$ is given by

$$\nabla g(\mathbf{a}, \mathbf{b}) = \nabla g(a_1, \ldots, a_{N-D}, b_1, \ldots, b_{N-D}) \triangleq \left(\frac{\partial g(\mathbf{a}, \mathbf{b})}{\partial a_1}, \ldots, \frac{\partial g(\mathbf{a}, \mathbf{b})}{\partial a_{N-D}}, \frac{\partial g(\mathbf{a}, \mathbf{b})}{\partial b_1}, \ldots, \frac{\partial g(\mathbf{a}, \mathbf{b})}{\partial b_{N-D}} \right) \tag{4}$$

The partial derivative of $g(\mathbf{a}, \mathbf{b})$ with respect to the i-th element of $\mathbf{a}$ is

$$\begin{aligned} \frac{\partial g(\mathbf{a}, \mathbf{b})}{\partial a_i} & = \frac{\partial}{\partial a_i} \left[\sum_{n=1}^{N-D} \left| a_n + jb_n - w_{\text{orig,mut},n} \right| \right] \\ & = \frac{\partial}{\partial a_i} \left| a_i + jb_i - w_{\text{orig,mut},i} \right| \\ & = \frac{\partial}{\partial a_i} \sqrt{\left(a_i - \text{Re}\left\{ w_{\text{orig,mut},i} \right\} \right)^2 + \left(b_i - \text{Im}\left\{ w_{\text{orig,mut},i} \right\} \right)^2} \\ & = \frac{1/2}{\sqrt{\left(a_i - \text{Re}\left\{ w_{\text{orig,mut},i} \right\} \right)^2 + \left(b_i - \text{Im}\left\{ w_{\text{orig,mut},i} \right\} \right)^2}} \frac{\partial}{\partial a_i} \left[\left(a_i - \text{Re}\left\{ w_{\text{orig,mut},i} \right\} \right)^2 + \left(b_i - \text{Im}\left\{ w_{\text{orig,} \right. \right. \right. \\ & = \frac{a_i - \text{Re}\left\{ w_{\text{orig,mut},i} \right\}}{\left| a_i + jb_i - w_{\text{orig,mut},i} \right|} \end{aligned}$$

The partial derivative of $g(\mathbf{a}, \mathbf{b})$ with respect to the i-th element of $\mathbf{b}$ is

$$
\begin{aligned}
\frac{\partial g(\mathbf{a},\mathbf{b})}{\partial b_i} &= \frac{\partial}{\partial b_i}\left[\sum_{n=1}^{N-D}\left|a_n + jb_n - w_{\text{orig,mut},n}\right|\right] \\
&= \frac{\partial}{\partial b_i}\left|a_i + jb_i - w_{\text{orig,mut},i}\right| \\
&= \frac{\partial}{\partial b_i}\sqrt{\left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)^2 + \left(b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}\right)^2} \\
&= \frac{1/2}{\sqrt{\left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)^2 + \left(b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}\right)^2}}\frac{\partial}{\partial b_i}\left[\left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)^2 + \left(b_i - \text{Im}\left\{w_{\text{orig,m}\right.\right.\right. \\
&= \frac{b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|}
\end{aligned}
$$

3.1.3 Hessian of the L1 cost function

The Hessian of $g(\mathbf{a},\mathbf{b})$ is a symmetric matrix given by

$$
H = \begin{bmatrix}
\frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial^2 a_1} & \cdots & \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial a_1 \partial a_{N-D}} & \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial a_1 \partial b_1} & \cdots & \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial a_1 \partial b_{N-D}} \\
 & \ddots & \vdots & \vdots & & \vdots \\
 & & \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial^2 a_{N-D}} & \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial a_{N-D} \partial b_1} & \cdots & \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial a_{N-D} \partial b_{N-D}} \\
 & & & \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial^2 b_1} & \cdots & \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial b_1 \partial b_{N-D}} \\
 & & & & \ddots & \vdots \\
 & & & & & \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial^2 b_{N-D}}
\end{bmatrix} \tag{5}
$$

The second derivative of $g(\mathbf{a},\mathbf{b})$ with respect to any one element of $\mathbf{a}$ is

$$
\begin{aligned}
\frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial^2 a_i} &= \frac{\partial}{\partial a_i}\frac{a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|} \\
&= \frac{\left|a_i + jb_i - w_{\text{orig,mut},i}\right| \frac{\partial}{\partial a_i}\left[a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right] - \left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)\frac{\partial}{\partial a_i}\left|a_i + jb_i - w_{\text{orig,mut},i}\right|}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^2} \\
&= \frac{\left|a_i + jb_i - w_{\text{orig,mut},i}\right| - \left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)\frac{a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|}}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^2} \\
&= \frac{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^2 - \left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)^2}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^3} \\
&= \frac{\text{Re}\left\{a_i + jb_i - w_{\text{orig,mut},i}\right\}^2 + \text{Im}\left\{a_i + jb_i - w_{\text{orig,mut},i}\right\}^2 - \left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)^2}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^3} \\
&= \frac{\left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)^2 + \left(b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}\right)^2 - \left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)^2}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^3}
\end{aligned}
$$

$$= \frac{\left(b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}\right)^2}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|^3}$$

The second derivative of $g(\mathbf{a}, \mathbf{b})$ with respect to any one element of $\mathbf{b}$ is

$$\begin{aligned}
\frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial^2 b_i} &= \frac{\partial}{\partial b_i} \frac{b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|} \\
&= \frac{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right| \frac{\partial}{\partial a_i}\left[b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}\right] - \left(b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}\right) \frac{\partial}{\partial a_i}\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|^2} \\
&= \frac{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right| - \left(b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}\right) \frac{b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|}}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|^2} \\
&= \frac{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|^2 - \left(b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}\right)^2}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|^3} \\
&= \frac{\mathrm{Re}\left\{a_i + jb_i - w_{\mathrm{orig,mut},i}\right\}^2 + \mathrm{Im}\left\{a_i + jb_i - w_{\mathrm{orig,mut},i}\right\}^2 - \left(b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}\right)^2}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|^3} \\
&= \frac{\left(a_i - \mathrm{Re}\left\{w_{\mathrm{orig,mut},i}\right\}\right)^2 + \left(b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}\right)^2 - \left(b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}\right)^2}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|^3} \\
&= \frac{\left(a_i - \mathrm{Re}\left\{w_{\mathrm{orig,mut},i}\right\}\right)^2}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|^3}
\end{aligned}$$

The mixed derivatives of $g(\mathbf{a}, \mathbf{b})$ with respect to any two elements of $\mathbf{a}$ are

$$\text{for } k \neq i, \quad \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial a_k \partial a_i} = \frac{\partial}{\partial a_k} \frac{a_i - \mathrm{Re}\left\{w_{\mathrm{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|} = 0 \tag{6}$$

The mixed derivatives of $g(\mathbf{a}, \mathbf{b})$ with respect to any two elements of $\mathbf{b}$ are

$$\text{for } k \neq i, \quad \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial b_k \partial b_i} = \frac{\partial}{\partial b_k} \frac{b_i - \mathrm{Im}\left\{w_{\mathrm{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|} = 0 \tag{7}$$

The mixed derivatives of $g(\mathbf{a}, \mathbf{b})$ with respect to any two one element of $\mathbf{a}$ and one of $\mathbf{b}$ with the same index is

$$\begin{aligned}
\frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial b_i \partial a_i} &= \frac{\partial}{\partial b_i} \frac{a_i - \mathrm{Re}\left\{w_{\mathrm{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|} \\
&= \left(a_i - \mathrm{Re}\left\{w_{\mathrm{orig,mut},i}\right\}\right) \frac{\partial}{\partial b_i} \frac{1}{\left|a_i + jb_i - w_{\mathrm{orig,mut},i}\right|}
\end{aligned}$$

$$
\begin{aligned}
&= \left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right) \frac{-1}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^2} \frac{\partial}{\partial b_i} \left|a_i + jb_i - w_{\text{orig,mut},i}\right| \\
&= \left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right) \frac{-1}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^2} \frac{b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|} \\
&= -\frac{\left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)\left(b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}\right)}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^3}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial b_i \partial a_i} &= \frac{\partial}{\partial a_i} \frac{b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|} \\
&= \left(b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}\right) \frac{\partial}{\partial a_i} \frac{1}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|} \\
&= \left(b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}\right) \frac{-1}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^2} \frac{\partial}{\partial a_i} \left|a_i + jb_i - w_{\text{orig,mut},i}\right| \\
&= \left(b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}\right) \frac{-1}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^2} \frac{a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|} \\
&= -\frac{\left(a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}\right)\left(b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}\right)}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|^3}
\end{aligned}
$$

The mixed derivatives of $g(\mathbf{a},\mathbf{b})$ with respect to any two one element of $\mathbf{a}$ and one of $\mathbf{b}$ with different indexes is

$$
\text{for } k \neq i, \quad \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial b_k \partial a_i} = \frac{\partial}{\partial b_k} \frac{a_i - \text{Re}\left\{w_{\text{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|} = 0 \tag{8}
$$

$$
\text{for } k \neq i, \quad \frac{\partial^2 g(\mathbf{a},\mathbf{b})}{\partial a_k \partial b_i} = \frac{\partial}{\partial a_k} \frac{b_i - \text{Im}\left\{w_{\text{orig,mut},i}\right\}}{\left|a_i + jb_i - w_{\text{orig,mut},i}\right|} = 0 \tag{9}
$$

3.2 1st and 2nd order derivatives of the constraint

In this section, an expression for the 1st order and 2nd order derivatives of the constraints defined is derived.

3.2.1 Inequality constraint for fmincon

Consider the inequality constraint that is verified when the corrected radiated field has a SLL lower than an arbitrary threshold for a given angular direction u_k.

$$|E_{\text{corr}}(u_k)|^2 \leq \text{SLL}_{\text{desired}} \, |E_{\text{corr}}(u_{ML})|^2 \tag{10}$$

The Matlab function `fmincon()` requires that, for each constraint $k \in \{1, \dots, K\}$, the quantity c_k is provided

$$c_k \triangleq |E_{\text{corr}}(u_k)|^2 - \text{SLL}_{\text{desired}} \, |E_{\text{corr}}(u_{ML})|^2 \tag{11}$$

Since the Matlab function `fmincon()` only works on real variables, the cost function will be derived with respect to the real and imaginary components of $\mathbf{w}_{\text{corr,mut}}$, $\mathbf{a}$ and $\mathbf{b}$, defined.

3.2.2 Gradient of the constraint

Let us separate the field radiated by the mutable and immutable elements. We denote with $\psi_{\text{mut},n}$ and $\psi_{\text{immut},n}$ are the embedded element factors for the n-th mutable and immutable element respectively

$$\psi_{\text{mut},n}(u) = e^{jkx_{\text{mut},n}u} \quad \text{and} \quad \psi_{\text{immut},n}(u) = e^{jkx_{\text{immut},n}u}, \tag{12}$$

and $x_{\text{mut},n}$ and $x_{\text{immut},n}$ are the positions of the n-th mutable and immutable elements respectively.

The radiated field can then be seen as a sum of the field radiated by the mutable elements and immutable elements

$$\begin{aligned} E_{\text{corr}}(u) &= \sum_{n=1}^{N} w_{\text{corr},n}\psi_n(u) \\ &= \sum_{n=1}^{N-D} w_{\text{corr,mut},n}\psi_{\text{mut},n}(u) + \sum_{n=1}^{D} w_{\text{corr,immut},n}\psi_{\text{immut},n}(u) \\ &= E_{\text{corr,mut}}(u) + E_{\text{corr,immut}}(u) \end{aligned}$$

where $E_{\text{corr,mut}}$ is the field radiated by the mutable elements with the corrected excitations

$$E_{\text{corr,mut}}(u) = \sum_{n=1}^{N-D} w_{\text{corr,mut},n}\psi_{\text{mut},n}(u_k) \tag{13}$$

and $E_{\text{corr,immut}}(u)$ is the field radiated by the immutable elements with the corrected excitations (that are equal to the faulty ones).

$$E_{\text{corr,immut}}(u) = \sum_{n=1}^{D} w_{\text{corr,immut},n} \psi_{\text{immut},n}(u_k) \tag{14}$$

The constraint function can then be rewritten as

$$c_k = \left| E_{\text{corr,mut}}(u_k) + E_{\text{corr,immut}}(u_k) \right|^2 - \text{SLL}_{\text{desired}} \left| E_{\text{corr,mut}}(u_{ML}) + E_{\text{corr,immut}}(u_{ML}) \right|^2$$

The gradient of the k-th constraint c_k with respect to the elements of **a** and **b** is defined as

$$\nabla c_k \triangleq \left(\frac{\partial c_k}{\partial a_1}, \ldots, \frac{\partial c_k}{\partial a_{N-D}}, \frac{\partial c_k}{\partial b_1}, \ldots, \frac{\partial c_k}{\partial b_{N-D}} \right). \tag{15}$$

As an intermediate step, the partial derivatives of $E_{\text{corr,mut}}(u)$ with respect to the i-th element of **a** and **b** will be derived

$$\begin{aligned} \frac{\partial E_{\text{corr,mut}}(u)}{\partial a_i} &= \frac{\partial}{\partial a_i} \left[\sum_{n=1}^{N-D} a_n \psi_{\text{mut},n}(u) + \sum_{n=1}^{N-D} j b_n \psi_{\text{mut},n}(u) \right] \\ &= \psi_{\text{mut},i}(u) \end{aligned}$$

$$\begin{aligned} \frac{\partial E_{\text{corr,mut}}(u)}{\partial b_i} &= \frac{\partial}{\partial a_i} \left[\sum_{n=1}^{N-D} a_n \overline{\psi_{\text{mut},n}(u)} + \sum_{n=1}^{N-D} j b_n \overline{\psi_{\text{mut},n}(u)} \right] \\ &= j \psi_{\text{mut},i}(u) \end{aligned}$$

Moreover, the partial derivatives of $\overline{E_{\text{corr}}(u)}$ with respect to the i-th element of **a** and **b** will be derived

$$\begin{aligned} \frac{\partial \overline{E_{\text{corr,mut}}(u)}}{\partial a_i} &= \frac{\partial}{\partial a_i} \left[\sum_{n=1}^{N-D} a_n \psi_{\text{mut},n}(u) + \sum_{n=1}^{N-D} j b_n \psi_{\text{mut},n}(u) \right] \\ &= \overline{\psi_{\text{mut},i}(u)} \end{aligned}$$

$$\begin{aligned} \frac{\partial \overline{E_{\text{corr,mut}}(u)}}{\partial b_i} &= \frac{\partial}{\partial a_i} \left[\sum_{n=1}^{N-D} a_n \overline{\psi_{\text{mut},n}(u)} + \sum_{n=1}^{N-D} j b_n \overline{\psi_{\text{mut},n}(u)} \right] \\ &= -j \overline{\psi_{\text{mut},i}(u)} \end{aligned}$$

Let us also compute the partial derivative of the squared magnitude of $\overline{E_{\text{corr}}(u)}$ with respect to the i-th element of **a** and **b** will be derived

$$\begin{aligned} \frac{\partial}{\partial a_i} \left| E_{\text{corr,mut}}(u_k) \right|^2 &= \frac{\partial}{\partial a_i} \left[E_{\text{corr,mut}}(u_k) \overline{E_{\text{corr,mut}}(u_k)} \right] \\ &= \overline{E_{\text{corr,mut}}(u_k)} \frac{\partial}{\partial a_i} E_{\text{corr,mut}}(u_k) + E_{\text{corr,mut}}(u_k) \frac{\partial}{\partial a_i} \overline{E_{\text{corr,mut}}(u_k)} \end{aligned}$$

$$
\begin{aligned}
&= \overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k) + E_{\text{corr,mut}}(u_k)\overline{\psi_{\text{mut},i}(u_k)} \\
&= \overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k) + \overline{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k)} \\
&= 2\text{Re}\left\{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial b_i}\left|E_{\text{corr,mut}}(u_k)\right|^2 &= \frac{\partial}{\partial b_i}\left[E_{\text{corr,mut}}(u_k)\overline{E_{\text{corr,mut}}(u_k)}\right] \\
&= \overline{E_{\text{corr,mut}}(u_k)}\frac{\partial}{\partial b_i}E_{\text{corr,mut}}(u_k) + E_{\text{corr,mut}}(u_k)\frac{\partial}{\partial b_i}\overline{E_{\text{corr,mut}}(u_k)} \\
&= j\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k) - jE_{\text{corr,mut}}(u_k)\overline{\psi_{\text{mut},i}(u_k)} \\
&= j\left(\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k) - \overline{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k)}\right) \\
&= -\frac{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k) - \overline{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k)}}{j} \\
&= -2\text{Im}\left\{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k)\right\}
\end{aligned}
$$

The following results will also be used in the derivation

$$
\begin{aligned}
\frac{\partial}{\partial a_i}\left[\left|E_{\text{corr,mut}}(u) + E_{\text{corr,immut}}(u)\right|^2\right] &= \frac{\partial}{\partial a_i}\left[\left|E_{\text{corr,mut}}(u)\right|^2 + 2\text{Re}\left\{E_{\text{corr,mut}}(u)\overline{E_{\text{corr,immut}}(u)}\right\} + \left|E_{\text{corr,immut}}(u)\right|^2\right] \\
&= \frac{\partial}{\partial a_i}\left[\left|E_{\text{corr,mut}}(u)\right|^2\right] + 2\text{Re}\left\{\overline{E_{\text{corr,immut}}(u)}\frac{\partial}{\partial a_i}\left[E_{\text{corr,mut}}(u)\right]\right\} \\
&= 2\text{Re}\left\{\overline{E_{\text{corr,mut}}(u)}\psi_{\text{mut},i}(u) + \overline{E_{\text{corr,immut}}(u)}\psi_{\text{immut},i}(u)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial b_i}\left[\left|E_{\text{corr,mut}}(u) + E_{\text{corr,immut}}(u)\right|^2\right] &= \frac{\partial}{\partial b_i}\left[\left|E_{\text{corr,mut}}(u)\right|^2 + 2\text{Re}\left\{E_{\text{corr,mut}}(u)\overline{E_{\text{corr,immut}}(u)}\right\} + \left|E_{\text{corr,immut}}(u)\right|^2\right] \\
&= \frac{\partial}{\partial b_i}\left[\left|E_{\text{corr,mut}}(u)\right|^2\right] + 2\text{Re}\left\{\overline{E_{\text{corr,immut}}(u)}\frac{\partial}{\partial b_i}\left[E_{\text{corr,mut}}(u)\right]\right\} \\
&= -2\text{Im}\left\{\overline{E_{\text{corr,mut}}(u)}\psi_{\text{mut},i}(u_k)\right\} + 2\text{Re}\left\{j\overline{E_{\text{corr,immut}}(u)}\psi_{\text{immut},i}(u)\right\} \\
&= -2\text{Im}\left\{\overline{E_{\text{corr,mut}}(u)}\psi_{\text{mut},i}(u_k)\right\} + 2\text{Re}\left\{j\text{Re}\left\{\overline{E_{\text{corr,immut}}(u)}\psi_{\text{immut},i}(u)\right\} - \text{Im}\left\{\overline{E_{\text{corr,immut}}(u)}\psi_{\text{immut},i}(u)\right\}\right\} \\
&= -2\text{Im}\left\{\overline{E_{\text{corr,mut}}(u)}\psi_{\text{mut},i}(u_k) + \overline{E_{\text{corr,immut}}(u)}\psi_{\text{immut},i}(u)\right\}
\end{aligned}
$$

The 1st order partial derivative of c_k with respect to the i-th element of $\mathbf{a}$ is given by

$$
\begin{aligned}
\frac{\partial c_k}{\partial a_i} &= \frac{\partial}{\partial a_i}\left[\left|E_{\text{corr,mut}}(u_k) + E_{\text{corr,immut}}(u_k)\right|^2 - \text{SLL}_{\text{desired}}\left|E_{\text{corr,mut}}(u_{ML}) + E_{\text{corr,immut}}(u_{ML})\right|^2\right] \\
&= 2\text{Re}\left\{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k) + \overline{E_{\text{corr,immut}}(u_k)}\psi_{\text{mut},i}(u_k)\right\} - 2\text{SLL}_{\text{desired}}2\text{Re}\left\{\overline{E_{\text{corr,mut}}(u_{ML})}\psi_{\text{mut},i}(u_{ML}) + \overline{E_{\text{corr,immut}}(u_{ML})}\psi_{\text{mut},i}(u_{ML})\right\} \\
&= 2\text{Re}\left\{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k) + \overline{E_{\text{corr,immut}}(u_k)}\psi_{\text{mut},i}(u_k) - 2\text{SLL}_{\text{desired}}\left(\overline{E_{\text{corr,mut}}(u_{ML})}\psi_{\text{mut},i}(u_{ML}) + \overline{E_{\text{corr,immut}}(u_{ML})}\psi_{\text{mut},i}(u_{ML})\right)\right\} \\
&= 2\text{Re}\left\{\overline{E_{\text{corr}}(u_k)}\psi_{\text{mut},i}(u_k) - \text{SLL}_{\text{desired}}\overline{E_{\text{corr}}(u_{ML})}\psi_{\text{mut},i}(u_{ML})\right\}
\end{aligned}
$$

The 1st order partial derivative of c_k with respect to the i-th element of $\mathbf{a}$ is given by

$$
\begin{aligned}
\frac{\partial c_k}{\partial b_i} &= \frac{\partial}{\partial b_i}\left[\left|E_{\text{corr,mut}}(u_k) + E_{\text{corr,immut}}(u_k)\right|^2 - \text{SLL}_{\text{desired}}\left|E_{\text{corr,mut}}(u_{ML}) + E_{\text{corr,immut}}(u_{ML})\right|^2\right] \\
&= -2\text{Im}\left\{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k) + \overline{E_{\text{corr,immut}}(u_k)}\psi_{\text{mut},i}(u_k)\right\} - \text{SLL}_{\text{desired}}\left(-2\text{Im}\left\{\overline{E_{\text{corr,mut}}(u_{ML})}\psi_{\text{mut},i}(u_{ML}) + \overline{E_{\text{corr,immut}}(u_{ML})}\psi_{\text{mut},i}(u_{ML})\right\}\right) \\
&= -2\text{Im}\left\{\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k) + \overline{E_{\text{corr,immut}}(u_k)}\psi_{\text{mut},i}(u_k) - \text{SLL}_{\text{desired}}\left(\overline{E_{\text{corr,mut}}(u_{ML})}\psi_{\text{mut},i}(u_{ML}) + \overline{E_{\text{corr,immut}}(u_{ML})}\psi_{\text{mut},i}(u_{ML})\right)\right\} \\
&= -2\text{Im}\left\{\overline{E_{\text{corr}}(u_k)}\psi_{\text{mut},i}(u_k) - \text{SLL}_{\text{desired}}\overline{E_{\text{corr}}(u_{ML})}\psi_{\text{mut},i}(u_{ML})\right\}
\end{aligned}
$$

3.2.3 Hessian of the constraint

The Hessian H_{c_k} of c_k with respect to $(\mathbf{a}, \mathbf{b})$ is a symmetric matrix given by

$$H_{c_k} = \begin{bmatrix} \frac{\partial^2 c_k}{\partial^2 a_1} & \cdots & \frac{\partial^2 c_k}{\partial a_1 \partial a_{N-D}} & \frac{\partial^2 c_k}{\partial a_1 \partial b_1} & \cdots & \frac{\partial^2 c_k}{\partial a_1 \partial b_{N-D}} \\ & \ddots & \vdots & \vdots & & \vdots \\ & & \frac{\partial^2 c_k}{\partial^2 a_{N-D}} & \frac{\partial^2 c_k}{\partial a_{N-D} \partial b_1} & \cdots & \frac{\partial^2 c_k}{\partial a_{N-D} \partial b_{N-D}} \\ & & & \frac{\partial^2 c_k}{\partial^2 b_1} & \cdots & \frac{\partial^2 c_k}{\partial b_1 \partial b_{N-D}} \\ & & & & \ddots & \vdots \\ & & & & & \frac{\partial^2 c_k}{\partial^2 b_{N-D}} \end{bmatrix} \quad (16)$$

To compute the Hessian, a few identities are used

$$\begin{aligned} \mathrm{Re}\left\{\frac{\partial}{\partial a_i}\left[\overline{E_{\mathrm{corr,mut}}(u)}\psi_{\mathrm{mut},l}(u)\right]\right\} &= \mathrm{Re}\left\{\psi_{\mathrm{mut},l}(u)\frac{\partial}{\partial a_i}\overline{E_{\mathrm{corr,mut}}(u)}\right\} \\ &= \mathrm{Re}\left\{\psi_{\mathrm{mut},l}(u)\overline{\psi_{\mathrm{mut},i}(u)}\right\} \\ &= \mathrm{Re}\left\{e^{jkx_{\mathrm{mut},l}u}e^{-jkx_{\mathrm{mut},i}u}\right\} \\ &= \mathrm{Re}\left\{e^{jk\left(x_{\mathrm{mut},l}-x_{\mathrm{mut},i}\right)u}\right\} \\ &= \cos\left(k\left(x_{\mathrm{mut},l}-x_{\mathrm{mut},i}\right)u\right) \end{aligned}$$

$$\begin{aligned} \mathrm{Im}\left\{\frac{\partial}{\partial b_i}\left[\overline{E_{\mathrm{corr,mut}}(u)}\psi_{\mathrm{mut},l}(u)\right]\right\} &= \mathrm{Im}\left\{\psi_{\mathrm{mut},l}(u)\frac{\partial}{\partial b_i}\overline{E_{\mathrm{corr,mut}}(u)}\right\} \\ &= \mathrm{Im}\left\{-j\psi_{\mathrm{mut},l}(u)\overline{y_{\mathrm{mut},i}(u)}\right\} \\ &= -\mathrm{Re}\left\{\psi_{\mathrm{mut},l}(u)\overline{y_{\mathrm{mut},i}(u)}\right\} \\ &= -\cos\left(k\left(x_{\mathrm{mut},l}-x_{\mathrm{mut},i}\right)u\right) \end{aligned}$$

The 2nd order mixed partial derivative of c_k with respect to the i-th element of $\mathbf{a}$ and the l-th element of $\mathbf{a}$ is given by

$$\begin{aligned} \frac{\partial^2 c_k}{\partial a_i \partial a_l} &= 2\mathrm{Re}\left\{\frac{\partial}{\partial a_i}\left[\overline{E_{\mathrm{corr,mut}}(u_k)}\psi_{\mathrm{mut},l}(u_k) - \mathrm{SLL_{desired}}\overline{E_{\mathrm{corr,mut}}(u_{ML})}\psi_{\mathrm{mut},l}(u_{ML})\right]\right\} \\ &= 2\mathrm{Re}\left\{\frac{\partial}{\partial a_i}\left[\overline{E_{\mathrm{corr,mut}}(u_k)}\psi_{\mathrm{mut},l}(u_k)\right]\right\} - \mathrm{SLL_{desired}}\mathrm{Re}\left\{\frac{\partial}{\partial a_i}\left[\overline{E_{\mathrm{corr,mut}}(u_{ML})}\psi_{\mathrm{mut},l}(u_{ML})\right]\right\} \\ &= 2\cos\left(k\left(x_{\mathrm{mut},l}-x_{\mathrm{mut},i}\right)u_k\right) - 2\mathrm{SLL_{desired}}\cos\left(k\left(x_{\mathrm{mut},l}-x_{\mathrm{mut},i}\right)u_{ML}\right) \end{aligned}$$

Similarly, 2nd order mixed partial derivative of c_k with respect to the i-th and l-th element of $\mathbf{b}$ is given by

$$\begin{aligned} \frac{\partial^2 c_k}{\partial b_i \partial b_l} &= -2\mathrm{Im}\left\{\frac{\partial}{\partial b_i}\left[\overline{E_{\mathrm{corr,mut}}(u_k)}\psi_{\mathrm{mut},l}(u_k) - \mathrm{SLL_{desired}}\overline{E_{\mathrm{corr,mut}}(u_{ML})}\psi_{\mathrm{mut},l}(u_{ML})\right]\right\} \\ &= -2\mathrm{Im}\left\{\frac{\partial}{\partial b_i}\left[\overline{E_{\mathrm{corr,mut}}(u_k)}\psi_{\mathrm{mut},l}(u_k)\right] - \mathrm{SLL_{desired}}\left[\overline{E_{\mathrm{corr,mut}}(u_{ML})}\psi_{\mathrm{mut},l}(u_{ML})\right]\right\} \end{aligned}$$

$$= \quad 2\cos\left(k\left(x_{\text{mut},l}-x_{\text{mut},i}\right)u_k\right)-2\text{SLL}_{\text{desired}}\cos\left(k\left(x_{\text{mut},l}-x_{\text{mut},i}\right)u_{ML}\right)$$

To compute the 2nd order mixed partial derivatives, the following identities are used

$$\begin{aligned}
\text{Re}\left\{\frac{\partial}{\partial b_l}\left[\overline{E_{\text{corr,mut}}(u)}\psi_{\text{mut},i}(u)\right]\right\} &= \text{Re}\left\{\psi_{\text{mut},i}(u)\frac{\partial}{\partial b_l}\overline{E_{\text{corr,mut}}(u)}\right\} \\
&= \text{Re}\left\{-j\psi_{\text{mut},i}(u)\overline{\psi_{\text{mut},l}(u)}\right\} \\
&= \text{Im}\left\{\psi_{\text{mut},i}(u)\overline{\psi_{\text{mut},l}(u)}\right\} \\
&= \text{Im}\left\{e^{jkx_{\text{mut},i}u}e^{-jkx_{\text{mut},l}u}\right\} \\
&= \text{Im}\left\{e^{jk\left(x_{\text{mut},i}-x_{\text{mut},l}\right)u}\right\} \\
&= \sin\left(k\left(x_{\text{mut},i}-x_{\text{mut},l}\right)u\right)
\end{aligned}$$

$$\begin{aligned}
\text{Im}\left\{\frac{\partial}{\partial a_i}\left[\overline{E_{\text{corr,mut}}(u)}\psi_{\text{mut},l}(u)\right]\right\} &= \text{Im}\left\{\psi_{\text{mut},l}(u)\frac{\partial}{\partial a_i}\overline{E_{\text{corr,mut}}(u)}\right\} \\
&= \text{Im}\left\{\psi_{\text{mut},l}(u)\overline{\psi_{\text{mut},i}(u)}\right\} \\
&= \sin\left(k\left(x_{\text{mut},i}-x_{\text{mut},l}\right)u\right) \\
&= -\sin\left(k\left(x_{\text{mut},l}-x_{\text{mut},i}\right)u\right)
\end{aligned}$$

The 2nd order mixed partial derivative of $c_k$with respect to the l-th element of $\mathbf{b}$ and the i-th element of $\mathbf{a}$ is given by

$$\begin{aligned}
\frac{\partial^2 c_k}{\partial b_l \partial a_i} &= 2\text{Re}\left\{\frac{\partial}{\partial b_l}\left[\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},i}(u_k)-\text{SLL}_{\text{desired}}\overline{E_{\text{corr,mut}}(u_{ML})}\psi_{\text{mut},i}(u_{ML})\right]\right\} \\
&= 2\sin\left(k\left(x_{\text{mut},i}-x_{\text{mut},l}\right)u_k\right)-2\text{SLL}_{\text{desired}}\sin\left(k\left(x_{\text{mut},i}-x_{\text{mut},l}\right)u_{ML}\right)
\end{aligned}$$

The 2nd order mixed partial derivative of $c_k$with respect to the i-th element of $\mathbf{a}$ and the l-th element of $\mathbf{b}$ is given by

$$\begin{aligned}
\frac{\partial^2 c_k}{\partial a_i \partial b_l} &= -2\text{Im}\left\{\frac{\partial}{\partial a_i}\left[\overline{E_{\text{corr,mut}}(u_k)}\psi_{\text{mut},l}(u_k)\right]-\text{SLL}_{\text{desired}}\frac{\partial}{\partial a_i}\left[\overline{E_{\text{corr,mut}}(u_{ML})}\psi_{\text{mut},l}(u_{ML})\right]\right\} \\
&= 2\sin\left(k\left(x_{\text{mut},i}-x_{\text{mut},l}\right)u_k\right)-2\text{SLL}_{\text{desired}}\sin\left(k\left(x_{\text{mut},i}-x_{\text{mut},l}\right)u_{ML}\right)
\end{aligned}$$

More information on the topics of this document can be found in the following list of references.

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