Introduction to Interval Arithmetic applied to Antenna Array Analysis

N. Anselmi, L. Manica, P. Rocca, A. Massa

Abstract

Errors and imperfections characterize almost every implementation of real physical devices. In an antenna array, elements' excitations values with are usually affected by errors due to imperfections on the feeding network components. This leads to implement an antenna radiating a pattern that generally differs from the expected one. In this framework, this report is aimed to introduce a novel methodology based on Interval Arithmetic aimed to calculate an upper and lower bounds on the power pattern given the interval tolerance of the amplifiers and of the phase shifters connected to the radiating elements. The power pattern features have been also defined exploiting the Interval Arithmetic tool, as well.

1 Interval Analysis

1.1 Real Interval Number (Interval)

A real interval number $\mathbf{X} = [a,\, b]$ or $[x] = [a,\, b]$ is the set $\{x,\, x \in \mathbb{R} |\, a \leq x \leq b\}$ of all real numbers x between and including the end points a (*infimum*) and b (*supremum*). Consequently a real number x is equivalent to an interval $[x,\, x]$. Such an interval is said to be *degenerate* interval. Moreover, the infimum a of the interval $\mathbf{X}$ is usually denoted as $\mathbf{X}_{inf}$ and the supremum of the interval $\mathbf{X}$ is usually denoted as $\mathbf{X}_{sup}$. An interval $\mathbf{X} = [a,\, b]$ is said to be *positive* (or *non-negative*) if $a \geq 0$, *strictly positive* if $a > 0$, *negative* (or *non-positive*) if $b \leq 0$, and *strictly negative* if $b < 0$. Two intervals $[a, b]$ and $[c, d]$ are *equal* if and only if $a = c$ and $b = d$. Intervals are partially ordered. $[a, b] < [c, d] \Leftrightarrow b < c$.

1.2 Interval Arithmetic

Interval arithmetic is the set of all operation on interval numbers: addition, subtraction, multiplication and division. Thus for a general operation $\circ \in \{+, -, \cdot, /\}$ the corresponding operation for intervals $\mathbf{X}$ and $\mathbf{Y}$ is

$$\mathbf{X} \circ \mathbf{Y} = \{x \circ y \,|\, x \in \mathbf{X},\ y \in \mathbf{Y}\} \tag{1}$$

Therefore the interval resulting from the operation contains every possible result of $\mathbf{X} \circ \mathbf{Y}$ for each $x \in \mathbf{X}$ and $y \in \mathbf{Y}$.

An interesting observation is that $\mathbf{X} \circ \mathbf{Y}$ can be represented by using only the bounds of $\mathbf{X} = [a, b]$ and $\mathbf{Y} = [c, d]$.

- **Interval Sum**

Let be $\mathbf{X} = [\mathbf{X}_{inf},\, \mathbf{X}_{sup}]$ and $\mathbf{Y} = [\mathbf{Y}_{inf},\, \mathbf{Y}_{sup}]$ two real intervals. The sum operation is defined as:

$$\mathbf{X} + \mathbf{Y} = [\mathbf{X}_{inf} + \mathbf{Y}_{inf},\, \mathbf{X}_{sup} + \mathbf{Y}_{sup}] \tag{2}$$

- **Interval Subtraction**

  Let be $\mathbf{X} = [\mathbf{X}_{inf},\, \mathbf{X}_{sup}]$ and $\mathbf{Y} = [\mathbf{Y}_{inf},\, \mathbf{Y}_{sup}]$ two real intervals. The subtraction operation is defined as:

$$\mathbf{X} - \mathbf{Y} = [\mathbf{X}_{inf} - \mathbf{Y}_{inf},\, \mathbf{X}_{sup} - \mathbf{Y}_{sup}] \tag{3}$$

- **Interval Multiplication**

  Let be $\mathbf{X} = [\mathbf{X}_{inf},\, \mathbf{X}_{sup}]$ and $\mathbf{Y} = [\mathbf{Y}_{inf},\, \mathbf{Y}_{sup}]$ two real intervals. The multiplication operation is defined as:

$$\begin{aligned} \mathbf{X} * \mathbf{Y} \quad &= \quad [\min\left(\mathbf{X}_{\mathrm{inf}}\mathbf{Y}_{\mathrm{inf}},\, \mathbf{X}_{\mathrm{inf}}\mathbf{Y}_{\mathrm{sup}},\, \mathbf{X}_{\mathrm{sup}}\mathbf{Y}_{\mathrm{inf}},\, \mathbf{X}_{\mathrm{sup}}\mathbf{Y}_{\mathrm{sup}}\right), \\ &\qquad \max\left(\mathbf{X}_{\mathrm{inf}}\mathbf{Y}_{\mathrm{inf}},\, \mathbf{X}_{\mathrm{inf}}\mathbf{Y}_{\mathrm{sup}},\, \mathbf{X}_{\mathrm{sup}}\mathbf{Y}_{\mathrm{inf}},\, \mathbf{X}_{\mathrm{sup}}\mathbf{Y}_{\mathrm{sup}}\right)] \end{aligned} \tag{4}$$

- **Interval Inverse**

  Let be$\mathbf{Y} = [\mathbf{Y}_{inf}, \mathbf{Y}_{sup}]$ a real interval. The inverse interval $1/\mathbf{Y}$ is defined as:

$$\frac{1}{\mathbf{Y}} = \left[\frac{1}{\mathbf{X}_{sup}}, \frac{1}{\mathbf{X}_{inf}}\right] 0 \notin \mathbf{Y} \tag{5}$$

- **Interval Division**

  Let be $\mathbf{X} = [\mathbf{X}_{inf}, \mathbf{X}_{sup}]$ and $\mathbf{Y} = [\mathbf{Y}_{inf}, \mathbf{Y}_{sup}]$ two real intervals. The interval division is defined by means of interval inverse and interval multiplication as:

$$\frac{\mathbf{X}}{\mathbf{Y}} = \mathbf{X} * \left(\frac{1}{\mathbf{Y}}\right) 0 \notin \mathbf{Y} \tag{6}$$

- **Power of a Real Interval**

  Let be $\mathbf{X} = [\mathbf{X}_{\text{inf}}, \mathbf{X}_{\text{sup}}]$ a real interval. The power of $\mathbf{X}$, $\mathbf{X}^n$ is a real interval computed as:

$$\mathbf{X}^n = \begin{cases} [1,\, 1] & if\, n = 0 \\ \left[\mathbf{X}_{\text{inf}}^n, \mathbf{X}_{\text{sup}}^n\right] & if\, \mathbf{X}_{\text{inf}} \geq 0\, or\, if\, \mathbf{X}_{\text{inf}} \leq 0 \leq \mathbf{X}_{\text{sup}}\, and\, n\, is\, odd \\ \left[\mathbf{X}_{\text{sup}}^n, \mathbf{X}_{\text{inf}}^n\right] & if\, \mathbf{X}_{\text{sup}} \leq 0 \\ \left[0,\, \max\left(\mathbf{X}_{\text{inf}}^n, \mathbf{X}_{\text{sup}}^n\right)\right] & if\, \mathbf{X}_{\text{inf}} \leq 0 \leq \mathbf{X}_{\text{sup}}\, and\, n\, is\, even \end{cases} \tag{7}$$

The set $I(\mathbb{R})$ of real compact intervals is closed with respect to these operations, this ensure that the result is a guaranteed inclusion of the solution obtained by any two values $x \in \mathbf{X}$ and $y \in \mathbf{Y}$.

Other useful intervals' definitions are:

- **Width of a Real Interval**

Let be $\mathbf{X} = [\mathbf{X}_{\text{inf}}, \mathbf{X}_{\text{sup}}]$ a real interval. The width of the interval $\mathbf{X}$, $w\left(\mathbf{X}\right)$ is a real number defined as:

$$w\left(\mathbf{X}\right) = \mathbf{X}_{\text{sup}} - \mathbf{X}_{\text{inf}} \tag{8}$$

- **Center of a Real Interval - Middle point**

  Let be $\mathbf{X} = [\mathbf{X}_{\text{inf}}, \mathbf{X}_{\text{sup}}]$ a real interval. The middle point of the interval $\mathbf{X}$, $m\left(\mathbf{X}\right)$ is defined as:

$$m\left(\mathbf{X}\right) = \frac{\mathbf{X}_{\text{sup}} - \mathbf{X}_{\text{inf}}}{2} \tag{9}$$

- **Equivalent representation of Intervals**

  Let be $\mathbf{X} = [\mathbf{X}_{\text{inf}}, \mathbf{X}_{\text{sup}}]$ a real interval, it can be represent equivalently in the form:

$$\mathbf{X} = \left[m\left(\mathbf{X}\right) - \frac{w\left(\mathbf{X}\right)}{2};\, m\left(\mathbf{X}\right) + \frac{w\left(\mathbf{X}\right)}{2}\right] \tag{10}$$

1.3 Complex Intervals

As for real numbers we can define intervals for the complex domain. The complex numbers are the field $\mathbb{C}$ of numbers of the form $x + i \cdot y$, where x, real component, and y, imaginary component, are real numbers and $i = \sqrt{-1}$ is the imaginary unit. This way of representation is called Cartesian form, with which a complex number can be represented as a point in the plane using the correspondence $x + i \cdot y \leftrightarrow (x, y)$.

An equivalent way to represent a complex is the so called polar form: $z = a \cdot e^{i \cdot \varphi}$, where polar coordinates modulo a and the phase φ, are used instead of the Cartesian ones. Using appropriate trigonometric formulas we can switch between the two representations obtaining the same point on the plane. This is not the case for complex intervals.

- **Complex Interval Cartesian Form (Rectangular Form)**

A complex intervals $\mathbf{X}$ is an ordered pair of intervals $\mathbf{X} = [\mathbf{A}, \mathbf{B}]$ with $\mathbf{A} = [\mathbf{A}_{inf}, \mathbf{A}_{sup}]$ and $\mathbf{B} = [\mathbf{B}_{inf}, \mathbf{B}_{sup}]$ real intervals. It consists in the set of complex numbers $x = a + i \cdot b$ such that:

$$\mathbf{X} = \{x \in \mathbb{C} \,|\, x = a + i \cdot b,\ \mathbf{A}_{\text{inf}} \leq a \leq \mathbf{A}_{\text{sup}}\, \mathbf{B}_{\text{inf}} \leq b \leq \mathbf{B}_{\text{sup}}\} \tag{11}$$

Graphically a complex interval in Cartesian form can be seen as a closed rectangular region in the complex plane, that's why it's also called rectangular complex interval.

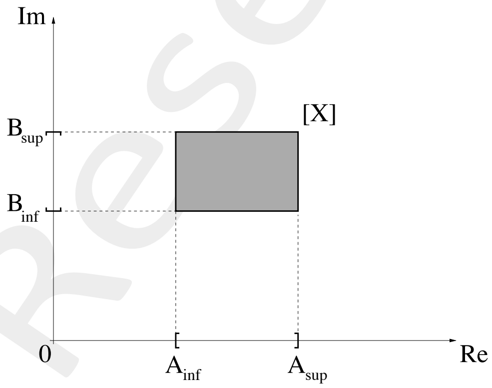

Figure 2.3.1: Complex interval, rectangular form

- **Complex Interval Polar Form (Sector)**

A polar complex interval $\mathbf{X}$ is a pair of intervals $\mathbf{X} = \{\mathbf{R}; \mathbf{\Theta}\}$ with $\mathbf{R} = [\mathbf{R}_{inf}, \mathbf{R}_{sup}]$ and $\mathbf{\Theta} = [\mathbf{\Theta}_{inf}, \Theta_{sup}]$ real intervals. It consists in the set of complex numbers $r \cdot e^{i\theta}$ such that:

$$\mathbf{X} = \{x \in \mathbb{C} \,|\, x = r \cdot e^{i\theta},\ r \in \mathbf{R},\ \theta \in \mathbf{\Theta}\} \tag{12}$$

In this case the corresponding area in the complex plane is not a rectangle but a bounded sector of a disk. We can see in Figure 2.3.2 that the two areas don't coincide: some complex numbers contained by the disk sector are not included into the rectangular area.

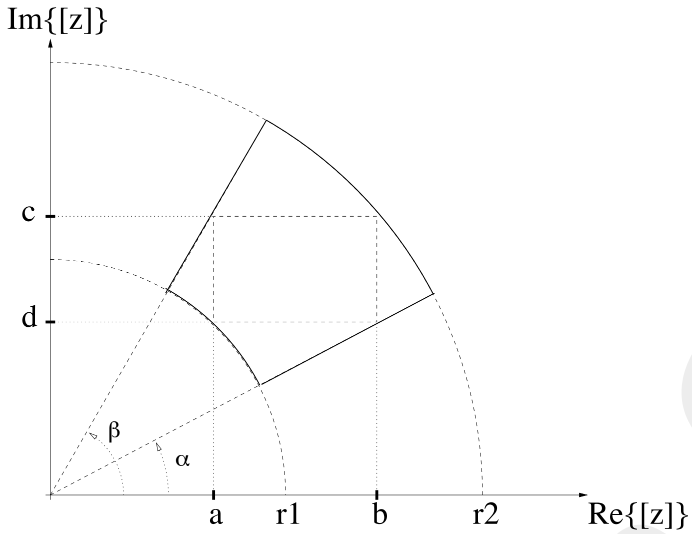

Figure 2.3.2: Complex interval in cartesian and polar form, $[z] = [a, b] + j \cdot [c, d]$ or $[z] = [r_1, r_2] \cdot e^{j \cdot [\alpha, \beta]}$.

1.4 Complex Interval Arithmetic

Arithmetic operations definition with complex intervals is not a straight and simple task as for real intervals. Unfortunately both complex interval representations cited above are not closed with respect to the arithmetic operations $\{+, -, \cdot, /\}$.

- **Sum and Difference of Rectangular Complex Intervals**

Handling complex intervals in cartesian form the addition and subtraction operations are exactly defined, the real and imaginary parts are treated in a similar way as in the real domain case. Let be $\mathbf{X} = [\mathbf{A}, \mathbf{A}']$ and $\mathbf{Y} = [\mathbf{B}, \mathbf{B}']$ two complex intervals. Using (2) the sum of the complex intervals is a complex interval defined as:

$$\mathbf{X} + \mathbf{Y} = [\mathbf{A} + \mathbf{B}, \mathbf{A}' + \mathbf{B}'] \tag{13}$$

Similarly using (3) the subtraction between $\mathbf{X}$ and $\mathbf{Y}$ can be defined as:

$$\mathbf{X} + \mathbf{Y} = [\mathbf{A} - \mathbf{B}, \mathbf{A}' - \mathbf{B}'] \tag{14}$$

- **Product and Division of Rectangular Complex Intervals**

Problems arise dealing with multiplications and divisions between complex intervals. The operations entails a rotation, thus the result must be wrapped in a rectangle, which introduces a good deal of pessimism. Anyway with rectangular intervals we can straightly compute the the final rounded result using equations (2), (3), (4), (6).

Let be $\mathbf{X} = [\mathbf{A}, \mathbf{A}']$ and $\mathbf{Y} = [\mathbf{B}, \mathbf{B}']$ two complex intervals. The product of the complex intervals is a complex interval defined as:

$$\mathbf{X} \cdot \mathbf{Y} = [\mathbf{A} \cdot \mathbf{B} - \mathbf{A}' \cdot \mathbf{B}', \mathbf{A} \cdot \mathbf{B}' + \mathbf{A}' \cdot \mathbf{B}] \tag{15}$$

The division of the complex intervals can defined using as:

$$\mathbf{X} \cdot \mathbf{Y} = \left[\frac{\mathbf{A} \cdot \mathbf{B} + \mathbf{A}' \cdot \mathbf{B}'}{\mathbf{B} \cdot \mathbf{B} + \mathbf{B}' \cdot \mathbf{B}'}, \frac{\mathbf{A}'\mathbf{B} - \mathbf{A}\mathbf{B}'}{\mathbf{B} \cdot \mathbf{B} + \mathbf{B}' \cdot \mathbf{B}'} \right] \tag{16}$$

We can notice that this expression is also affected by the so called *dependency effect*, further briefly explained, having the same interval occurring more than once in the same computation.

2 Variables Definition

Let me firstly introduce some definitions about antenna array theory:

- **Array Factor**

For a linear array the array factor is defined as:

$$AF(\theta) = \sum_{n=1}^{N} w_n e^{j\beta d_n u} \tag{17}$$

where $\beta = \frac{2\pi}{\lambda}$, $u = sin(\theta)$ with $\theta \in [-\pi/2; \pi/2]$, and $d_n;\ n = 1, \ldots, N$ is the position of the $n-th$ element on the array axis. For a uniform linear array $d_n = (n-1)$. w_n is the n-element's excitation coefficient.

- **Excitation Coefficients**

$$w_n = a_n e^{j\varphi_n} \tag{18}$$

where $a_n;\ n = 1, \ldots, N$ are the amplitudes and $\varphi_n;\ n = 1, \ldots, N$ are the phases of the $w_n;\ n = 1, \ldots, N$ excitation coefficients.

- **Dynamic Range Ratio**

$$DRR = \frac{max\{a_n\}}{min\{a_n\}}\ n = 1, ..., N \tag{19}$$

- **Power Pattern**

The power pattern radiated by a linear array is:

$$P(\theta) = AF(\theta) \cdot AF(\theta)^*;\ \theta \in [-\pi/2; \pi/2] \tag{20}$$

where $AF(\theta)^*$is the complex conjugate of the array factor. It can also be expressed in an equivalent way:

$$P(\theta) = Re\{AF(\theta)\}^2 + Im\{AF(\theta)\}^2;\ \theta \in [-\pi/2; \pi/2] \tag{21}$$

Here some useful definitions for interval analysis:

- **Interval Excitation Coefficients**

$$[w_n] = [a_n] e^{j[\varphi_n]} \tag{22}$$

where

$$[a_n] = [a_n - \delta a_n, a_n + \delta a_n] \qquad with\ \delta a_n = K \cdot a_n;\ 0 < K < 1 \tag{23}$$

$$[\varphi_n] = [\varphi_n - \delta\varphi_n, \varphi_n + \delta\varphi_n] \qquad with\ \delta\varphi_n = \xi;\ \xi \in \mathbb{R}^+ \tag{24}$$

It's worth noting that the tolerance depends by the nominal value, in this case $\delta a_n = K \cdot a_n$as shown in Figure:

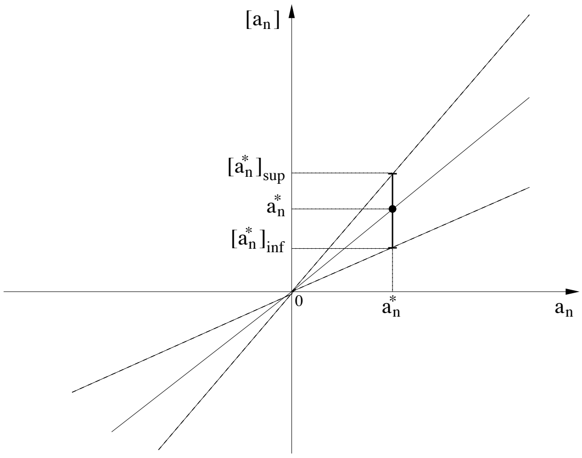

Figure 3.1. Amplitude interval versus nominal value

- **Interval Dynamic Range Ratio**

$$[DRR] = [DRR_{inf}, \, DRR_{sup}] \tag{25}$$

where DRR_{inf} and DRR_{sup} are defined as:

$$DRR_{inf} = \frac{max\{[a_n]_{inf}\}}{min\{[a_n]_{sup}\}} \tag{26}$$

$$DRR_{sup} = \frac{max\{[a_n]_{sup}\}}{min\{[a_n]_{inf}\}} \tag{27}$$

2.1 Interval Array Factor

Now considering purely real excitations, $[AF(u)]$ is defined as:

$$[AF(u)] = \sum_{n=1}^{N} [a_n] e^{j\beta d_n u} \tag{28}$$

Rewriting equation (28) we obtain:

$$[AF(u)] = \sum_{n=1}^{N} [a_n] \cdot cos(\beta d_n u) + j \cdot [a_n] \cdot sin(\beta d_n u) \tag{29}$$

Now after some interval arithmetic we can define:

$$Re\{[AF(u)]\}_{sup} = \sum_{n=1}^{N} a_n \cdot cos(\beta d_n u) + \delta a_n \cdot |cos(\beta d_n u)| \tag{30}$$

$$Re\{[AF(u)]\}_{inf} = \sum_{n=1}^{N} a_n \cdot cos(\beta d_n u) - \delta a_n \cdot |cos(\beta d_n u)| \tag{31}$$

$$Im\{[AF(u)]\}_{sup} = \sum_{n=1}^{N} a_n \cdot sin(\beta d_n u) + \delta a_n \cdot |sin(\beta d_n u)| \tag{32}$$

$$Im\{[AF(u)]\}_{inf} = \sum_{n=1}^{N} a_n \cdot sin(\beta d_n u) - \delta a_n \cdot |sin(\beta d_n u)| \tag{33}$$

and remembering that $\delta a_n = K \cdot a_n$, the respectively interval widths are:

$$w(Re\{[AF(u)]\}) = 2K \cdot \sum_{n=1}^{N} a_n \cdot |cos(\beta d_n u)| \tag{34}$$

$$w(Im\{[AF(u)]\}) = 2K \cdot \sum_{n=1}^{N} a_n \cdot |sin(\beta d_n u)| \tag{35}$$

Now using (10) the interval array factor can be expressed as:

$$\begin{aligned} [AF(u)] \quad = \quad & \left[Re\{AF(u)\} - \tfrac{w(Re\{[AF(u)]\})}{2},\, Re\{AF(u)\} + \tfrac{w(Re\{[AF(u)]\})}{2}\right] + \\ & j \cdot \left[Im\{AF(u)\} - \tfrac{w(Im\{[AF(u)]\})}{2},\, Im\{AF(u)\} + \tfrac{w(Im\{[AF(u)]\})}{2}\right] \end{aligned} \tag{36}$$

2.2 Interval Power Pattern

Now let us derive a suitable formulation for the Interval Power Pattern. From equation (21) we can straightly define:

$$[P(\theta)] = [Re\{[AF(\theta)]\}^2 + Im\{[AF(\theta)]\}^2];\ \theta \in [-\pi/2; \pi/2] \tag{37}$$

using the redefined interval power operation with $n = 2$ (7), we can compute the $Re\{[AF(\theta)]\}^2$ and $Im\{[AF(\theta)]\}^2$.

Below the derived final formulas for $[P(u)]_{inf}$and $[P(u)]_{sup}$are reported:

- **Interval Power Pattern's Supremum**

As regards $[P(u)]_{sup}$the derived formula is:

$$\begin{aligned} [P(u)]_{sup} &= Re\left\{AF\left(\theta\right)\right\}^{2} + \frac{w\left(Re\left\{\left[AF\left(\theta\right)\right]\right\}\right)^{2}}{4} + Im\left\{AF\left(\theta\right)\right\}^{2} + \frac{w\left(Im\left\{\left[AF\left(\theta\right)\right]\right\}\right)^{2}}{4} + \\ &+ \left|Re\left\{AF\left(\theta\right)\right\}\right| \cdot w\left(Re\left\{\left[AF\left(\theta\right)\right]\right\}\right) + \left|Im\left\{AF\left(\theta\right)\right\}\right| \cdot w\left(Im\left\{\left[AF\left(\theta\right)\right]\right\}\right) \end{aligned} \quad (38)$$

- **Interval Power Pattern's Infimum**

Computing $[P(u)]_{inf}$ we need to discriminate four different cases:

- if $\left(Re\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\}, Re\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}\right) \leq 0$ $\left(Re\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\}, Re\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}\right) \geq 0$ and $\left(Im\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\}, Im\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}\right) \leq 0$ $\left(Im\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\}, Im\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}\right) \geq 0$

$$\begin{aligned} [P(u)]_{inf} &= Re\left\{AF\left(\theta\right)\right\}^{2} + \frac{w\left(Re\left\{\left[AF\left(\theta\right)\right]\right\}\right)^{2}}{4} + Im\left\{AF\left(\theta\right)\right\}^{2} + \frac{w\left(Im\left\{\left[AF\left(\theta\right)\right]\right\}\right)^{2}}{4} + \\ &- \left|Re\left\{AF\left(\theta\right)\right\}\right| \cdot w\left(Re\left\{\left[AF\left(\theta\right)\right]\right\}\right) - \left|Im\left\{AF\left(\theta\right)\right\}\right| \cdot w\left(Im\left\{\left[AF\left(\theta\right)\right]\right\}\right) \end{aligned}$$

- if $\left(Re\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\}, Re\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}\right) \leq 0$ $\left(Re\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\}, Re\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}\right) \geq 0$ and $Im\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\} \leq 0 \leq Im\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}$

$$[P(u)]_{inf} = Re\left\{AF\left(\theta\right)\right\}^{2} + \frac{w\left(Re\left\{\left[AF\left(\theta\right)\right]\right\}\right)^{2}}{4} - \left|Re\left\{AF\left(\theta\right)\right\}\right| \cdot w\left(Re\left\{\left[AF\left(\theta\right)\right]\right\}\right)$$

- if $Re\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\} \leq 0 \leq Re\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}$ and $\left(Im\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\}, Im\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}\right) \leq 0$ $\left(Im\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\}, Im\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}\right) \geq 0$

$$[P(u)]_{inf} = Im\left\{AF\left(\theta\right)\right\}^{2} + \frac{w\left(Im\left\{\left[AF\left(\theta\right)\right]\right\}\right)^{2}}{4} - \left|Im\left\{AF\left(\theta\right)\right\}\right| \cdot w\left(Im\left\{\left[AF\left(\theta\right)\right]\right\}\right)$$

- if $Re\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\} \leq 0 \leq Re\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}$ and $Im\left\{\left[AF\left(\theta\right)\right]_{\text{inf}}\right\} \leq 0 \leq Im\left\{\left[AF\left(\theta\right)\right]_{\text{sup}}\right\}$

$$[P(u)]_{inf} = 0$$

3 Interval Array Pattern Parameters

In this section interval version of the main array parameters are defined. Directivity, SLL, and HPBW are some important parameters in the design of antenna arrays; in the following is explained how to compute them for linear equi-spaced arrays.

3.1 Interval Directivity

The formula here presented is valid for linear array with N isotropic elements, spaced by $\frac{\lambda}{2}$.

Suppose to dispose the N array elements along the z-axis starting from the origin:

The array factor is

$$AF(u) = \sum_{n=0}^{N-1} [w_n] e^{jkdu} \tag{39}$$

with $k = \frac{2\pi}{\lambda}$, $d = \frac{\lambda}{2}$, $u = cos(\theta)$. $[w_n]$ are the complex interval weights.

Directivity is defined as:

$$D(\theta, \phi) = \frac{U(\theta, \phi)}{\frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} U(\theta, \phi) sin(\theta) d\theta d\phi} \tag{40}$$

where $U(\theta, \phi)$ is the radiation intensity:

$$U(\theta, \phi) = \frac{r^2}{2\eta} |\underline{E}_{tot}(\underline{r}, \theta, \phi)|^2 = U_0(\theta, \phi) |AF(\theta, \phi)|^2 \tag{41}$$

$U_0(\theta, \phi)$ is radiation intensity for the single element. In our case, considering ideal isotropic sources, $U_0(\theta, \phi) = 1$. Usually "peak directivity" is more used, that is the directivity calculated at its maximum: $D_{max}(\theta_0, \phi_0)$. For our array, radiating at bore-sight, the direction of the maximum is at $\theta = \frac{\pi}{2}$; thus equation (23) reduces to:

$$U(\theta_0, \phi_0) = |AF(\theta_0, \phi_0)|^2 = |\sum_{n=0}^{N-1} w_n e^{jkdncos(\theta_0)}|^2 = |\sum_{n=0}^{N-1} w_n|^2 \tag{42}$$

Now, computing the denominator of (22), we obtain:

$$\frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} U(\theta, \phi) sin(\theta) d\theta d\phi = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} w_n w_m^* \int_{-1}^{1} e^{jkd(n-m)u} du \tag{43}$$

which for $d = \frac{\lambda}{2}$ reduces to $\sum_{n=0}^{N-1} |w_n|^2$, obtaining the expression of peak directivity:

$$D_{max} = \frac{|\sum_{n=0}^{N-1} w_n|^2}{\sum_{n=0}^{N-1} |w_n|^2} \tag{44}$$

valid for half-wavelength-spaced isotropic elements.

Now converting the expression above in an interval form:

$$[D_{max}] = \frac{|\sum_{n=0}^{N-1}[w_n]|^2}{\sum_{n=0}^{N-1}|[w_n]|^2} \tag{45}$$

3.2 Side Lobe Level for Interval Arrays

It's important to define a proper interval version for the significant figure of merit SLL. In our case we can derive from the interval radiation pattern, two different beam patterns:

$$|AF(u)|^2_{inf} = inf\{|[AF(u)]|^2\} \tag{46}$$

$$|AF(u)|^2_{sup} = sup\{|[AF(u)]|^2\} \tag{47}$$

Now I can define the interval SLL as $[SLL] = [SLL_{inf}, SLL_{sup}]$ where:

$$SLL_{inf} = -[max_{u \in ML}\{|AF(u)|^2_{sup}\} - max_{u \in SL}\{|AF(u)|^2_{inf}\}] \tag{48}$$

$$SLL_{sup} = -[max_{u \in ML}\{|AF(u)|^2_{inf}\} - max_{u \in SL}\{|AF(u)|^2_{sup}\}] \tag{49}$$

where ML and SL are the sets of u corresponding to the Main Lobe and the Side Lobes regions.

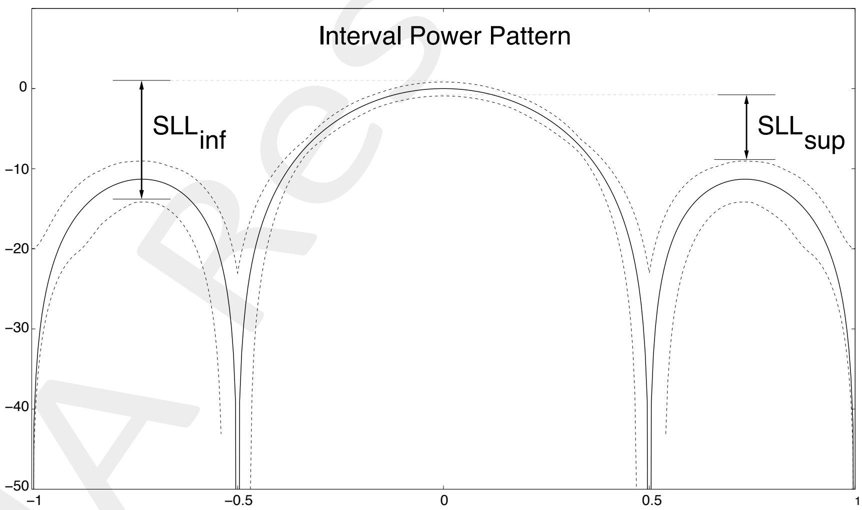

Figure 4.2.1: Worst and best SLL example.

3.3 Half Power Beam Width

Also in this case I can define the interval HPBW, as $[HPBW] = [HPBW_{inf}, HPBW_{best}]$ where $HPBW_{inf}$ and $HPBW_{sup}$ are respectively described in Figure 4.3.1 and in Figure 4.3.2.

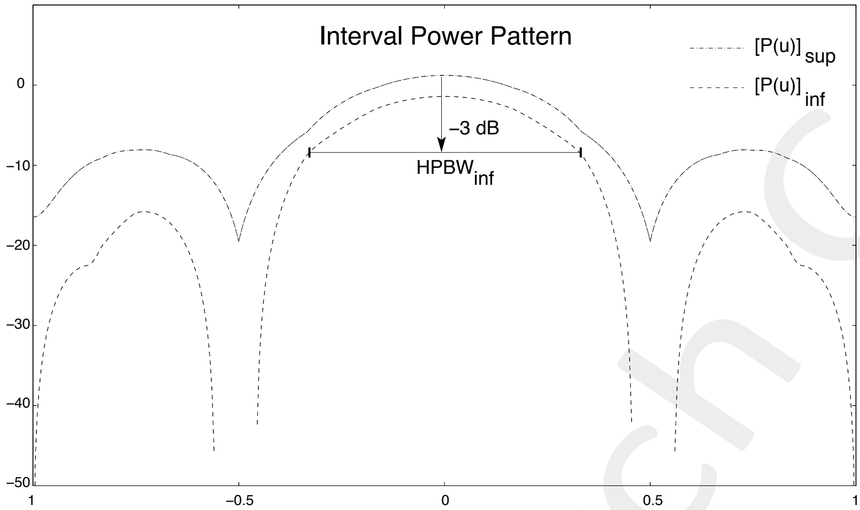

Figure 4.3.1: Infimum of [HPBW]

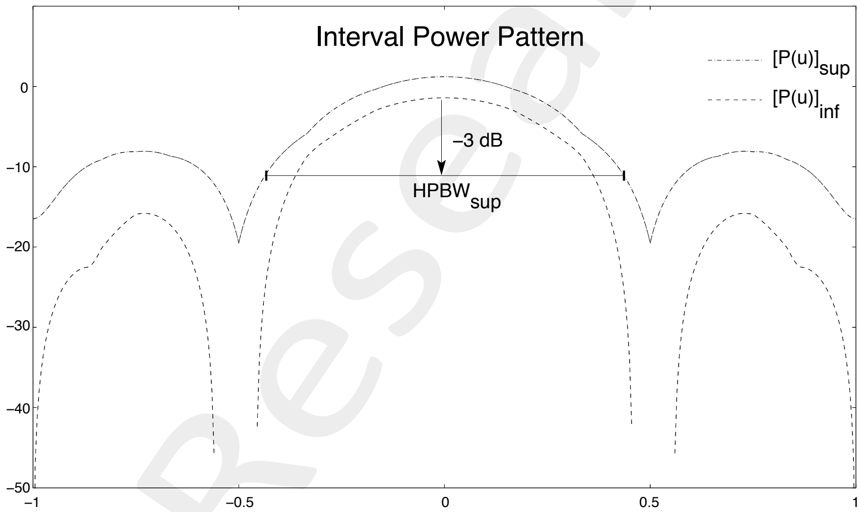

Figure 4.3.2 : Supremum of [HPBW]

3.4 Pattern Tolerance

The Pattern Tolerance parameter $\triangle$ measures the area between the $|AF(u)|^2_{sup}$ and $|AF(u)|^2_{inf}$ and is defined as:

$$\triangle = \int_{-1}^{1} (|AF(u)|^2_{sup} - |AF(u)|^2_{inf}) du \tag{50}$$

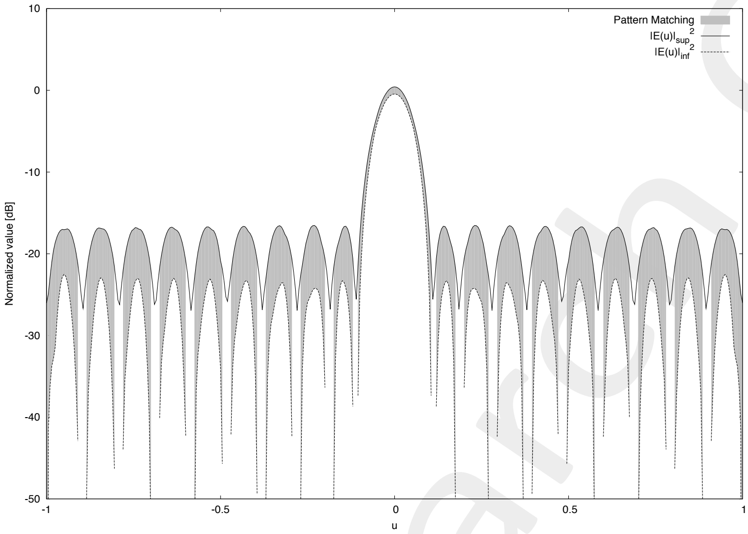

Figure 4.4.1: Pattern Tolerance

- **Normalized Pattern Tolerance**

The Normalized Pattern tolerance parameter Δ_{norm} measures the area between the $|AF(u)|^2_{sup}$ and $|AF(u)|^2_{inf}$ divided by the radiated power $|AF(u)|^2$:

$$\Delta_{norm} = \frac{\int_{-1}^{1} (|AF(u)|^2_{sup} - |AF(u)|^2_{inf}) du}{\int_{-1}^{1} |AF(u)|^2 du} \tag{51}$$

- **Peak Interval**

This parameter measure the gap between the infimum and the supremum radiation patterns at the peak of them.

$$[|AF(u_{max})|^2] = [|AF(u_{max})|^2_{inf}, \, |AF(u_{max})|^2_{sup}] \tag{52}$$

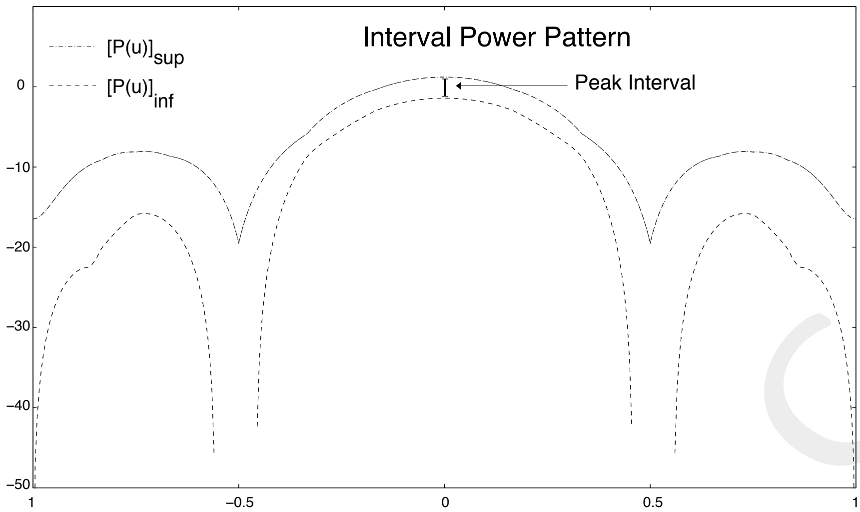

Figure 4.4.2: Peak Interval Parameter

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