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Abstract

The present contribution deals with ray propagation in semi-infinite percolation lattices consisting of a succession of uniform density layers. The problem of analytically evaluating the probability that a single ray penetrates up to a prescribed level before being reflected back into the above empty half-plane is addressed. A hybrid approach, exploiting the complementarity of two mathematical models in dealing with uniform configurations, is presented and assessed through numerical ray-tracing-based experiments in order to show improvements upon previous predictions techniques.

Key words:

Stochastic ray tracing, Stratified random media, Percolation theory.

1 Introduction

This paper deals with wave propagation in stratified random lattices where the electromagnetic source is external to the half-plane filled by the obstacles and it radiates a plane monochromatic wave impinging on the lattice with a known incidence angle θ. By assuming that the dimension of each site is large with respect to the wavelength, the wave is modeled in terms of a collection of parallel rays that undergo specular reflections on occupied cells. The aim is analytically estimating the probability $\Pr\{0 \longmapsto k\}$ that a single ray reaches a prescribed level k inside the lattice before being reflected back in the above empty half-plane.

The problem concerned with a uniform random grid where sites are occupied with a known probability $q = 1 - p$ was addressed in [1]. Ray propagation was modeled in terms of a one-dimensional stochastic process and $\Pr\{0 \longmapsto k\}$ was evaluated by applying the theory of the Martingale random processes [2] and the so-called Wald's approximation. The extension to the two-dimensional case, where an isotropic source is located inside the random lattice, has been proposed in [3]. Moreover, the dual case of small obstacles has been dealt with in [4] and [5].

As far as the one-dimensional percolation model is concerned, the approach proposed in [1] and referred to as Martingale approach (MTGA), has been successively extended to the nonuniform case, where the occupancy probability changes according to a known obstacles' density $q(j)$, j being the row index [6]. In order to apply the theory of the Martingale random processes and the Wald's approximation, the ray jumps following the first one are assumed independent, identically-distributed, with mean and standard deviation approaching zero. Both mathematical considerations and numerical experiments have shown that these conditions are verified provided that (*a*) the incidence angle is not far from 45^o or a large number of reflections takes place, (*b*) the percolation lattice in hand is dense and (*c*) the density profile does not present abrupt changes in value between adjacent levels and a significant variation along the lattice. With reference to the last condition, it is evident that the MTGA fails when dealing with ray propagation in stratified random lattices, since such configurations are characterized by step-like variations in the density profile. In order to overcome such a drawback, an ad hoc formulation, referred to in the following as Multi-layer Martingale approach (MMTGA), has been described in [7]. Starting from the observation that a stratified random grid is made up by a succession of uniform density layers, the propagation inside each single layer is faithfully described by the

MTGA [but still under conditions (*a*) and (*b*)]. Thus, mathematically binding the terms predicting propagation in each single uniform layer leads to a formal description of the ray propagation in the whole stratified lattice.

Another approach for to estimating the probability $\Pr\{0 \longmapsto k\}$ is the so-called Markov approach (MKVA) [8], where the ray propagation is modeled by means of a Markov chain [9]. By observing that whenever a ray hits an occupied vertical face it does not change its vertical direction of propagation, only reflections on occupied horizontal faces, whose number is independent from the incidence angle θ, play a relevant role in evaluating $\Pr\{0 \longmapsto k\}$. Likewise the MTGA, the MKVA properly works provided that some conditions are verified. In particular, it looses accuracy, when (*i*) the incidence angle deviates from 45^o and (*ii*) the obstacles' density increases. However, there are no requirements on the density profile and thus, when stratified random lattices are taken into account, the MKVA allows reliable predictions [under conditions (*i*) and (*ii*)] and a customized formulation is not needed .

A comparison between the MMTGA and the MKVA when dealing with ray propagation in stratified random lattices has been presented in [10]. Numerical experiments have pointed out that the MMTGA outperforms the MKVA when dense lattices are considered, while the MKVA offers more reliable predictions when the stratified grid is constituted by sparse layers. Such results suggest that it could be profitable to consider a hybrid procedure exploiting in a complementary fashion the MTGA [1] and the MKVA [8].

The paper is organized as follows. In Section 2, the problem is stated and the mathematical formulation is briefly resumed. Section 3 deals with the numerical validation, showing improvements of the proposed approach upon previous results. Final comments and conclusions are drawn in Section 4.

2 Problem Statement and Mathematical Formulation

Let us consider a stratified random grid (Fig. 1) made up by a succession of uniform layers denoted by the index n. The n-th layer is made up by L_n levels characterized by the same occupancy probability q_n and identified by the relative index $i = 1, ..., L_n$. Accordingly, each single level inside the grid is identified by $l_{n,i} = j = i + \sum_{t=1}^{n-1} L_t$ and the density profile is mathematically described in terms of

$$q(l_{n,i}) = q_n; \;\; n = 1, 2, ...; \; i = 1, ..., L_n. \tag{1}$$

Ray propagation along the whole lattice is described by means of the Markov chain depicted in Figure 2, where states j^+ and j^- denote a ray traveling inside the level j with positive and negative direction, respectively. Such a schema allows one to mathematically bind state-of-the-art *building blocks* denoting the probability that a ray freely crosses layer n (i.e., the probability that a ray, traveling with positive direction in the level $l_{n,1}$, reaches level l_{n,L_n} before being reflected back in level $l_{n,1}$, $P_n \hat{=} \Pr\left\{l^+_{n,1} \longmapsto l^+_{n,L_n} \prec l^-_{n,1}\right\}$). The mutual exclusive event is referred to as $Q_n = 1 - P_n \hat{=} \Pr\left\{l^+_{n,1} \longmapsto l^-_{n,1} \prec l^+_{n,L_n}\right\}$. Moreover, it is worth noting that, due to symmetry, the probability $\Pr\left\{l^-_{n,L_n} \longmapsto l^-_{n,1} \prec l^+_{n,L_n}\right\}$ that a ray freely crosses layer n traveling with negative direction is equal to P_n, and $\Pr\left\{l^-_{n,L_n} \longmapsto l^+_{n,L_n} \prec l^-_{n,1}\right\} = Q_n$. Now, let us focus on the probabilities of transition between adjacent layers, i.e., $\Pr\left\{l^+_{n,L_n} \longmapsto l^+_{n+1,1} \prec l^-_{n,L_n}\right\}$ and $\Pr\left\{l^-_{n+1,1} \longmapsto l^-_{n,L_n} \prec l^-_{n+1,1}\right\}$. As far as the first term is concerned, a ray traveling with positive direction through level l_{n,L_n} reaches next layer $(n+1)$, keeping its direction of propagation, with probability p_{n+1} (i.e., the probability that the horizontal face between layer n and layer $n+1$ is free). Such an event holds true whatever the number of reflections on vertical faces occurring at level l_{n,L_n}, since they do not change the vertical direction of propagation of the ray. Similar reasoning leads to $\Pr\left\{l^-_{n+1,1} \longmapsto l^-_{n,L_n} \prec l^-_{n+1,1}\right\} = p_n$.

Now, with reference to the Markov chain model and by assuming that level k belongs to the K-th layer, i.e., $l_{K,1} \le k \le l_{K,L_K}$, the following closed form relation is obtained

$$\Pr\{0 \longmapsto k\} = \frac{p_1}{\frac{1}{P_1} + p_1 \sum_{n=2}^{K} \left[\frac{1-P_n}{p_n P_n} + \frac{q_n}{p_n p_{n-1}}\right]}. \tag{2}$$

The building blocks P_n, $n = 1, ..., K$, can be evaluated either by means of the MKVA [8],

$$P_n = P_n^{(MKVA)} = \frac{p_n}{1 + (L_n - 2) q_n}, \tag{3}$$

or through the MTGA [1],

$$P_n = P_n^{(MTGA)} = \begin{cases} 1, & i=1, \\ \frac{p_n}{q_{e_n}} \left[\frac{1-p_{e_n}^{(L_n-1)}}{L_n-1} \right], & i>1, \end{cases} \tag{4}$$

where $p_{e_n} = 1 - q_{e_n} = p_n^{\tan\theta+1}$ is the effective probability that a ray crosses a level with occupancy probability q_n without any reflections and $L_K = (k - l_{K,1})$. The core idea of this paper is to fully exploit the complementarity of the MKVA and of the MTGA in describing ray propagation in uniform random lattices. For each uniform layer of the grid, a choice between (3) and (4) is performed on the basis of the incidence condition and of the obstacles' density of the layer at hand. To rigorously define the choice criterion (i.e., the range of q_n and θ values such that one approach rather than the other is more suitable to be applied) numerical experiments, reported in the next section, have been carried out.

In passing, we observe that a-priori applying either (3) or (4) along the whole lattice (i.e., $\forall n$, $n = 1, ..., K$), independently from q_n and θ, leads to the approaches compared in [10], i.e., the MKVA and the MMTGA, respectively.

3 Numerical Validation

In this section, selected numerical results, assessing the effectiveness of the hybrid solution compared with the MMTGA and the MKVA, are reported. As a reference solution, the penetration probability $\Pr\{0 \longmapsto k\}$ has been numerically estimated by means of computer-based ray tracing experiments performed according to the procedure described in [1]. In order to quantify the prediction accuracy, let us define the *prediction error*

$$\varepsilon_k \triangleq \frac{|\Pr_S\{0 \longmapsto k\} - \Pr\{0 \longmapsto k\}|}{\max_k [\Pr_S\{0 \longmapsto k\}]} \times 100, \tag{5}$$

where the sub-script S indicates numerically-computed values, and the *mean error*

$$\langle\varepsilon\rangle \triangleq \frac{\sum_{k=1}^{k_{MAX}} \varepsilon_k}{k_{MAX}}, \tag{6}$$

k_{MAX} being the total number of levels in the numerical experiment at hand. Moreover, in order to analyze the mean behavior when N density profiles are considered, let us define the *global mean error*

$$\langle \phi \rangle \triangleq \frac{\sum_{i=1}^{N} \langle \varepsilon \rangle_i}{N}, \tag{7}$$

where $\langle \varepsilon \rangle_i$ is the mean error relative to the i-th profile.

3.1 Calibration

The aim of this section is to figure out a choice criterion according to that either the MKVA (3) or the MTGA (4) is selected as building block P_n, $n = 1, ..., K$, in (2). Towards this end, the uniform density profiles obtained by varying q between 0.05 and 0.4[(1)] with a step of 0.05, have been considered. Moreover, different incidence conditions have been taken into account, namely $\theta = \{15^o, 30^o, 45^o, 60^o, 75^o\}$ and $\Pr\{0 \longmapsto k\}$ has been evaluated in the first $k_{MAX} = 10$ levels.

With reference to the obtained mean error values (Fig. 3), the following rule has been stated:

- if $q_n < 0.2$ [Figs. 3(*a*)-(*c*)], then $P_n = P_n^{(MKVA)}$ whatever θ;
- if $0.2 \leq q_n \leq 0.3$ [Figs. 3(*d*)-(*f*)], then $P_n = P_n^{(MTGA)}$ if $\theta = 30^o$ and $P_n = P_n^{(MKVA)}$ elsewhere;
- if $q_n > 0.3$ [Figs. 3(*g*)-(*h*)], then $P_n = P_n^{(MKVA)}$ if $\theta = 15^o$ and $P_n = P_n^{(MTGA)}$ otherwise.

3.2 Numerical Assessment and Comparisons

This section is aimed at assessing the proposed approach, referred to in the following as hybrid approach (HYBA), by means of an exhaustive numerical validation. In particular, an analysis on the role of the problem parameters in affecting the estimation accuracy, along with a comparison with the MKVA and the MMTGA, will be presented by considering different test cases. Three- and four-layers profiles having $L_n = 8$, $n = 1, ..., K-1$ and $L_K = k_{MAX} - 8(K-1)$,

(1) Higher q values have been not taken into account, since for values higher than the so-called percolation threshold q_c ($q_c \approx 0.40725$ for the two-dimensional case) the propagation is inhibited [11].

obtained by taking into account all the possible combinations of the occupancy probability values $q_n = \{0.05,\, 0.15,\, 0.25,\, 0.35\}$, $n = 1, ..., K$, have been considered. Moreover, different impinging directions, $\theta = \{15^o,\, 30^o,\, 45^o,\, 60^o,\, 75^o\}$, have been assumed and the penetration probability has been estimated in the first $k_{MAX} = 32$ levels.

Firstly, let us analyze the behavior of the global mean error $\langle\phi\rangle$ for different θ values. With reference to Figure 4, by comparing the plots concerned with three- and four-layers profiles, it can be observed that the number of layers K does not affect the prediction accuracy of the considered approaches. In particular, concerning the HYBA, $|\langle\phi\rangle_{4L} - \langle\phi\rangle_{3L}| \leq 0.04$ whatever θ, where subscripts "$4L$" and "$3L$" refer to three- and the four-layers profiles, respectively. As far as the dependence on the incidence angle is concerned, it is evident that the accuracy of the HYBA (as well as that of the other approaches) increases as θ tends to 45^o. This is fully predictable, since both the building blocks (3) and (4) ensure the best performances when $\theta = 45^o$ [8]. Finally, it turns out that on average the HYBA outperforms both the MKVA and the MMTGA whatever θ and in a more significant fashion when $\theta = 75^o$ ($\left[\frac{\langle\phi\rangle_{MMTGA}}{\langle\phi\rangle_{HYBA}}\right] \cong 2.5$ and $\left[\frac{\langle\phi\rangle_{MKVA}}{\langle\phi\rangle_{HYBA}}\right] \cong 1.8$).

The second test case deals with two four-layers profiles. The former, indicated as "$4L_S$", is made up by sparse layers (i.e., $q_1 = q_3 = 0.05$, $q_2 = q_4 = 0.15$), while the latter, "$4L_D$", is dense (i.e., $q_1 = q_3 = 0.25$, $q_2 = q_4 = 0.35$). By analyzing Table I, where the mean error values when $\theta = 45^o$ are reported, it can be observed that, as expected, the HYBA satisfactorily performs in both cases ($\left[\frac{\langle\varepsilon\rangle_{4L_D}}{\langle\varepsilon\rangle_{4L_S}}\right]_{HYBA} \cong 1$), while the MKVA and the MMTGA are sensitive to the obstacles' density, allowing more reliable predictions in correspondence with profile "$4L_S$" ($\left[\frac{\langle\varepsilon\rangle_{4L_D}}{\langle\varepsilon\rangle_{4L_S}}\right]_{MKVA} = 2$) and "$4L_D$" ($\left[\frac{\langle\varepsilon\rangle_{4L_S}}{\langle\varepsilon\rangle_{4L_D}}\right]_{MMTGA} \cong 1.7$), respectively.

The last test case deals with a four-layers profile consisting of very sparse and very dense layers in alternated succession (i.e., $q_1 = q_3 = 0.05$, $q_2 = q_4 = 0.35$). The plots in Figure 5, as well as the $\langle\varepsilon\rangle$ values reported in Tab. II, point out that the HYBA outperforms both the MKVA and the MMTGA whatever θ. The effectiveness of the HYBA is more evident when $\theta = 75^o$ [Fig. 7(*e*) and last row of Tab. II] as confirmed by the following indexes $\frac{\langle\varepsilon\rangle_{MKVA}}{\langle\varepsilon\rangle_{HYBA}} = 4.11$ and $\frac{\langle\varepsilon\rangle_{MMTGA}}{\langle\varepsilon\rangle_{HYBA}} = 4.22$. A final observation is concerned with the role of the variation size in the occupation probability value between adjacent layers, i.e., $\Delta_{n,n+1} = |q_{n+1} - q_n|$. By considering the case $\theta = 45^o$ and comparing the $\langle\varepsilon\rangle$ values of the test case in hand ($\Delta_{n,n+1} = 0.3$) with those of the previous test case ($\Delta_{n,n+1} = 0.1$, Tab. I), it can be observed that performances are not affected by $\Delta_{n,n+1}$. Such an event points out the reliability of the Markov chain model in correctly predicting the change in the slope of $\Pr\{0 \longmapsto k\}$ that occurs in correspondence with the border between adjacent layers.

4 Conclusions

In this paper, a hybrid formulation for predicting the ray propagation in stratified random lattices has been presented. The proposed solution exploits the positive features of the Martingale approach and the Markov approach in dealing with uniform random grids, compensating at the same time their drawbacks. Numerical experiments have demonstrated the effectiveness and the reliability of the proposed solution as well as the improvements with respect to other stochastic approaches. Summarizing, the following considerations can be drawn:

- Both the number of layers and the variation in the occupation probability value between adjacent layers do not affect performances. Such a behavior assesses the effectiveness of the Markov chain model (Fig. 2) and the relative solution (2) in modeling propagation in the whole stratified lattice.

- Unlike the MMTGA and the MKVA, the performances of the hybrid method are not affected by the obstacles' density.

- The hybrid technique outperforms the other methods, whatever the incidence angle and the obstacles' density.

- Whatever the approach, the most reliable predictions are obtained when $\theta = 45^o$.

Acknowledgments

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Figure Captions

- **Figure 1.** Example of a propagating ray in a stratified lattice. The grid is a realization of the obstacles' density distribution reported on the left-hand side.

- **Figure 2.** Markov chain modeling the ray propagation inside a stratified random lattice.

- **Figure 3.** Uniform profiles - Mean error $\langle \varepsilon \rangle$ versus θ when (*a*) $q = 0.05$, (*b*) $q = 0.10$, (*c*) $q = 0.15$, (*d*) $q = 0.20$, (*e*) $q = 0.25$, (*f*) $q = 0.30$, (*g*) $q = 0.35$, and (*h*) $q = 0.40$.

- **Figure 4.** Multi-layers profiles - Global mean error $\langle \phi \rangle$ versus θ for (*a*) the three-layers profiles and (*b*) the four-layers profiles.

- **Figure 5.** Multi-layers profiles - $\Pr \{0 \longmapsto k\}$ versus k for a four-layers profile with $q_1 = q_3 = 0.05$ and $q_2 = q_4 = 0.35$ when (*a*) $\theta = 15^o$, (*b*) $\theta = 30^o$, (*c*) $\theta = 45^o$, (*d*) $\theta = 60^o$, and (*e*) $\theta = 75^o$.

Table Captions

- **Table I.** Mean error $\langle \varepsilon \rangle$ for four-layers profiles "$4L_S$" ($q_1 = q_3 = 0.05$ and $q_2 = q_4 = 0.15$) and "$4L_D$" ($q_1 = q_3 = 0.25$ and $q_2 = q_4 = 0.35$) when $\theta = 45^o$.

- **Table II.** Mean error $\langle \varepsilon \rangle$ for a four-layers profile characterized by $q_1 = q_3 = 0.05$ and $q_2 = q_4 = 0.35$.

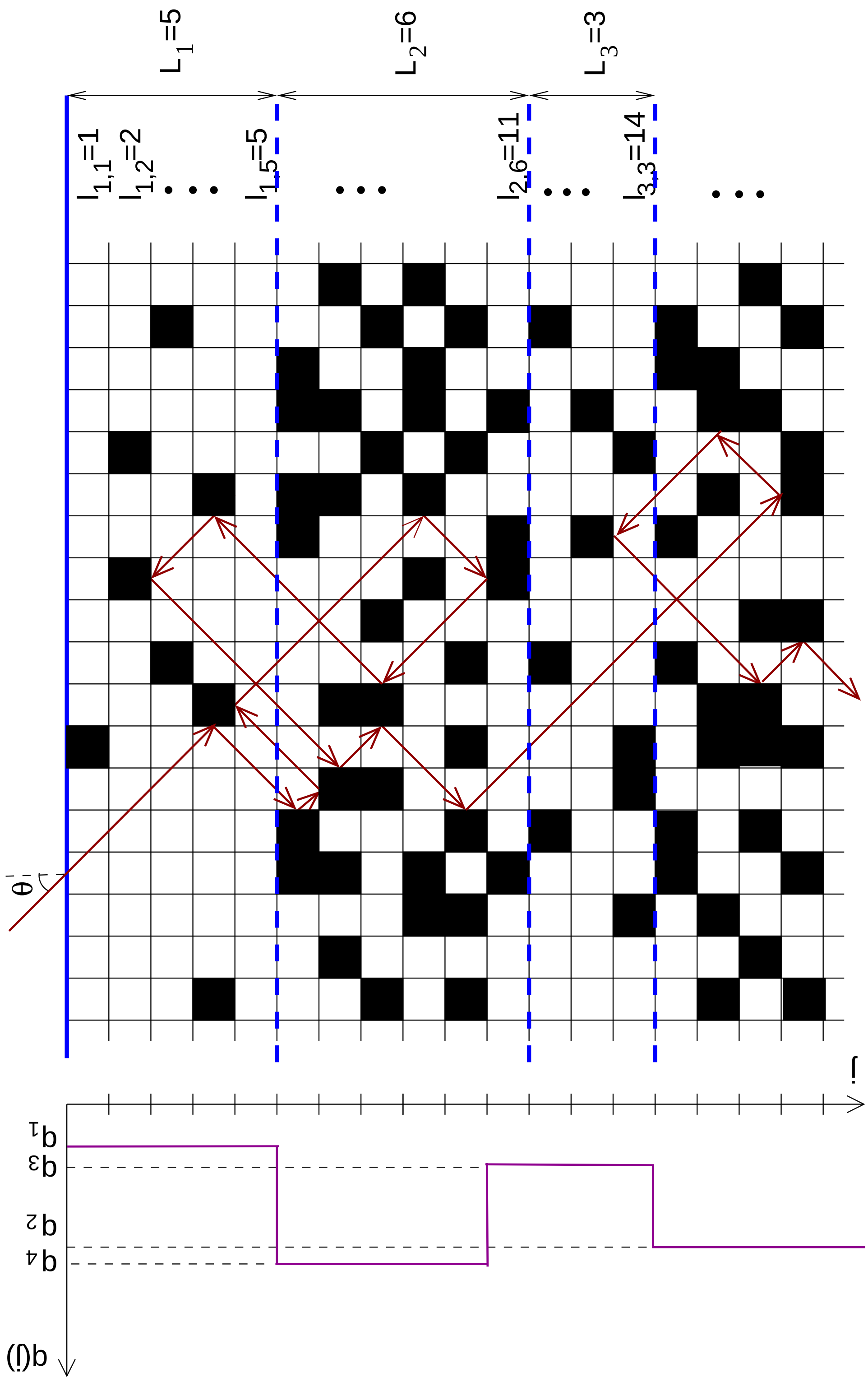


Fig. 1 - A. Martini *et al.*, ”A Hybrid Approach for Modeling Stochastic ...”

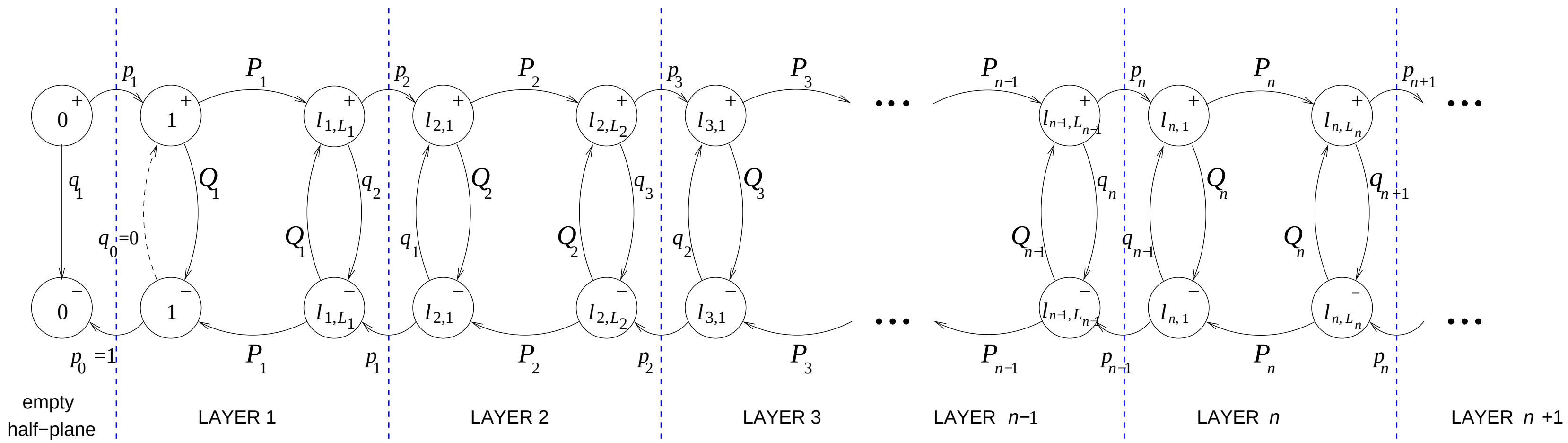


Fig. 2 - A. Martini *et al.*, ”A Hybrid Approach for Modeling Stochastic ...”

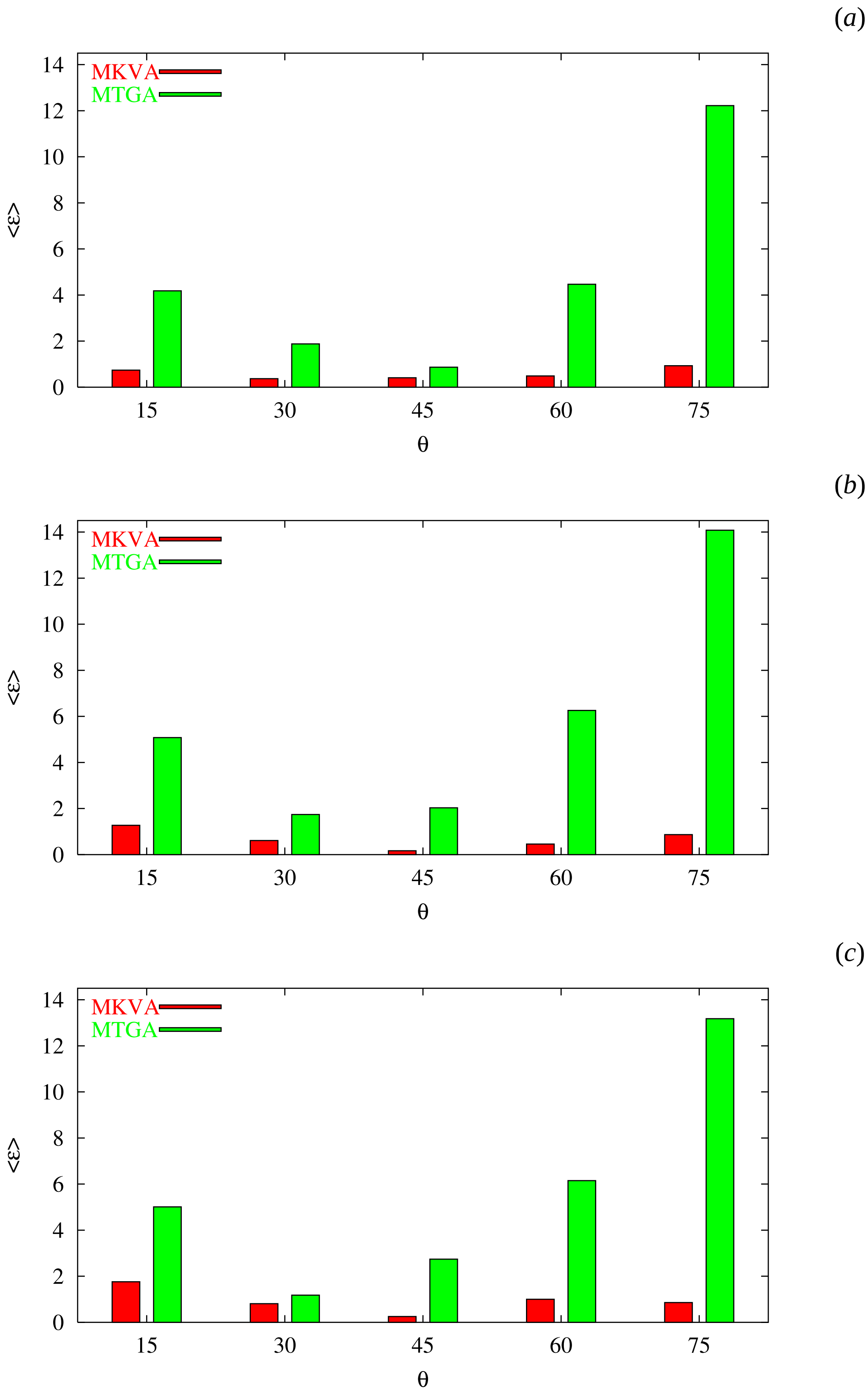
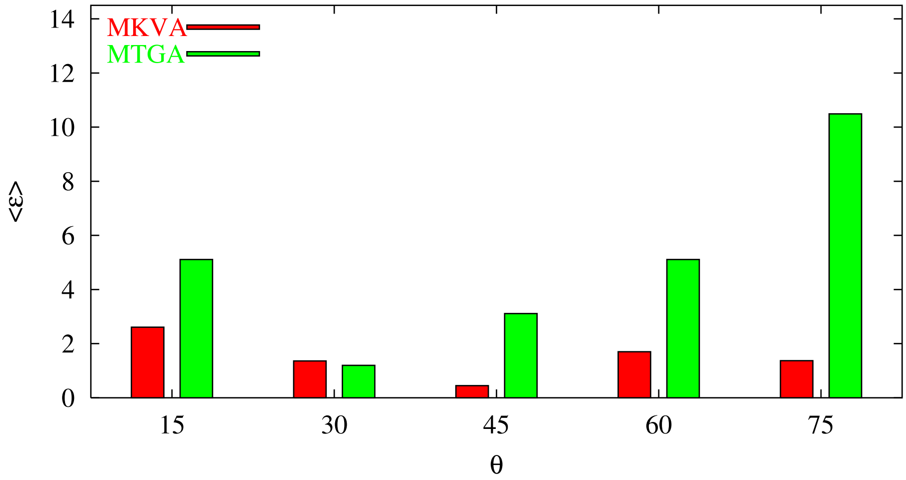


Fig. 3 (*I*) - A. Martini *et al.*, ”A Hybrid Approach for Modeling Stochastic ...”

(*d*)



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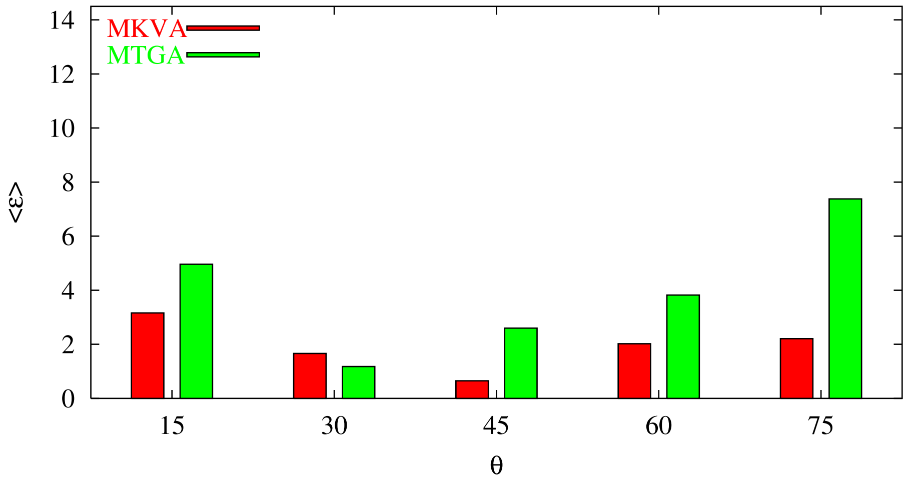


(*f*)

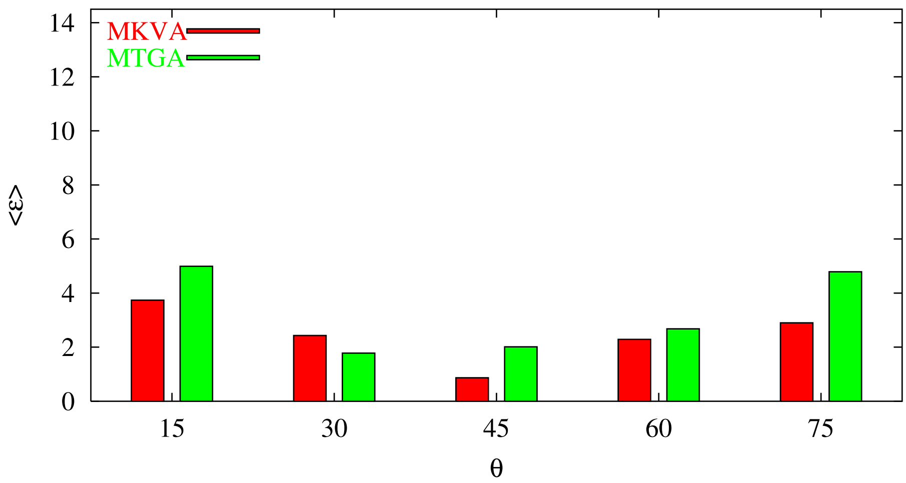
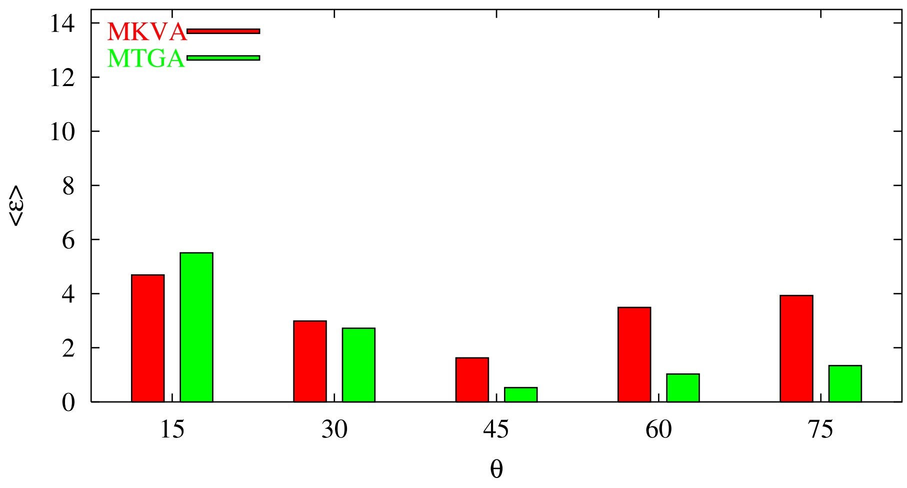


Fig. 3 (*II*) - A. Martini *et al.*, "A Hybrid Approach for Modeling Stochastic ..."

(*g*)



(*h*)

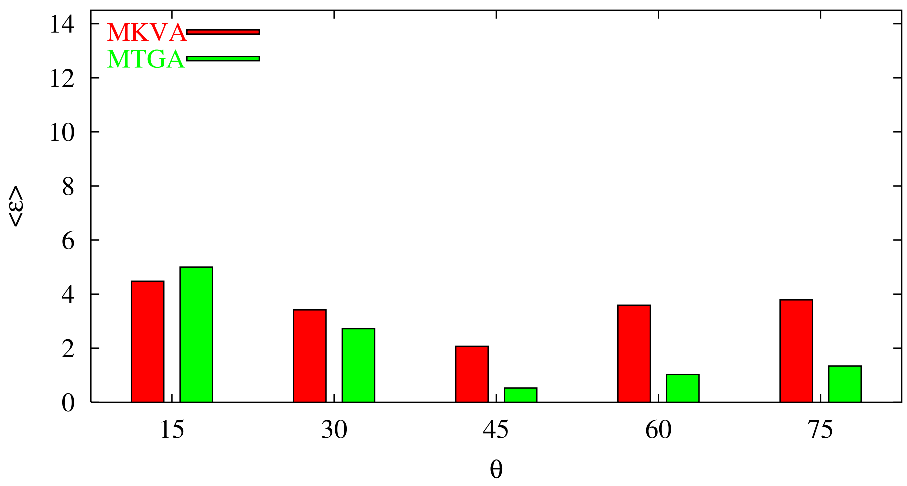
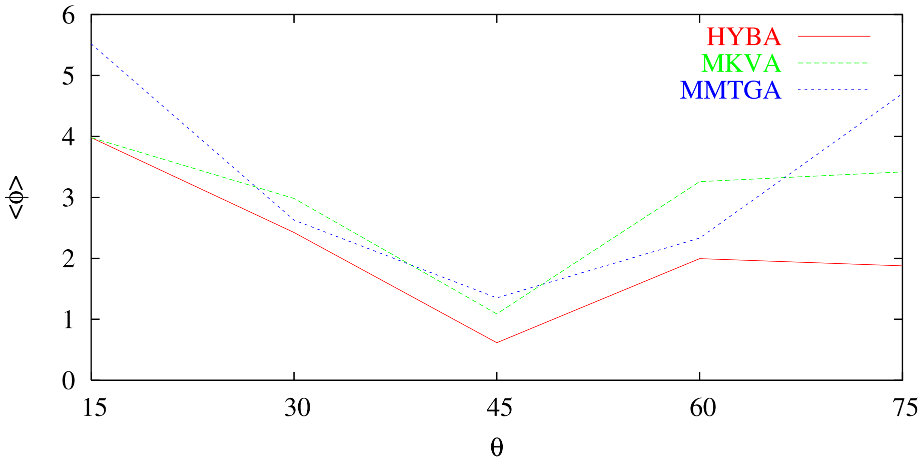


Fig. 3 (*III*) - A. Martini *et al.*, ”A Hybrid Approach for Modeling Stochastic ...”

(a)



(b)

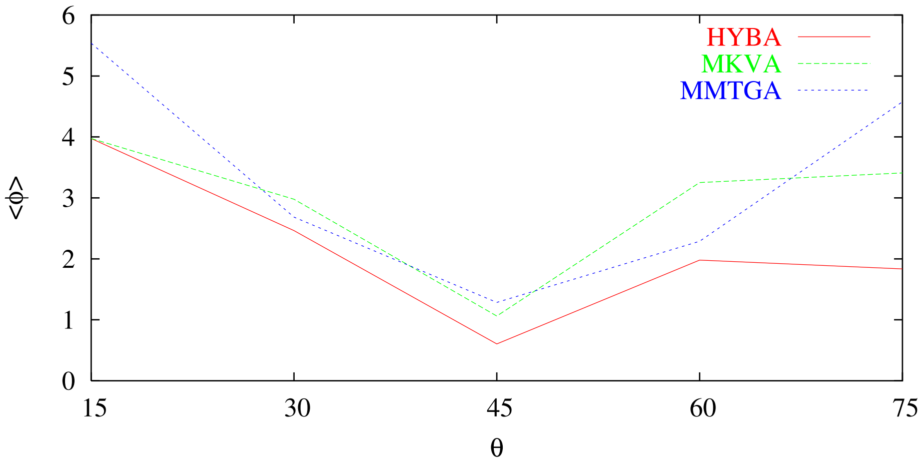


Fig. 4 - A. Martini *et al.*, ”A Hybrid Approach for Modeling Stochastic ...”

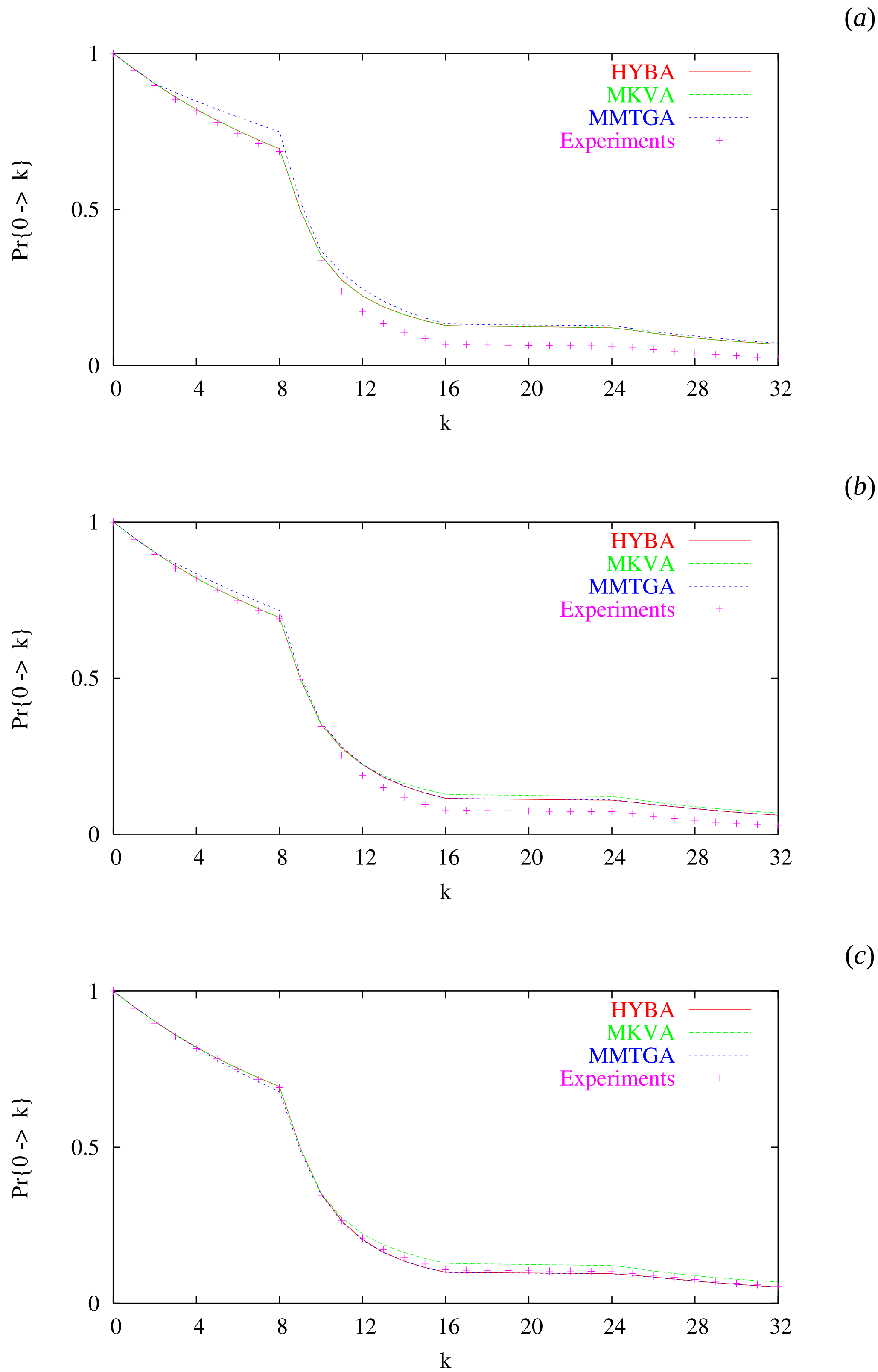
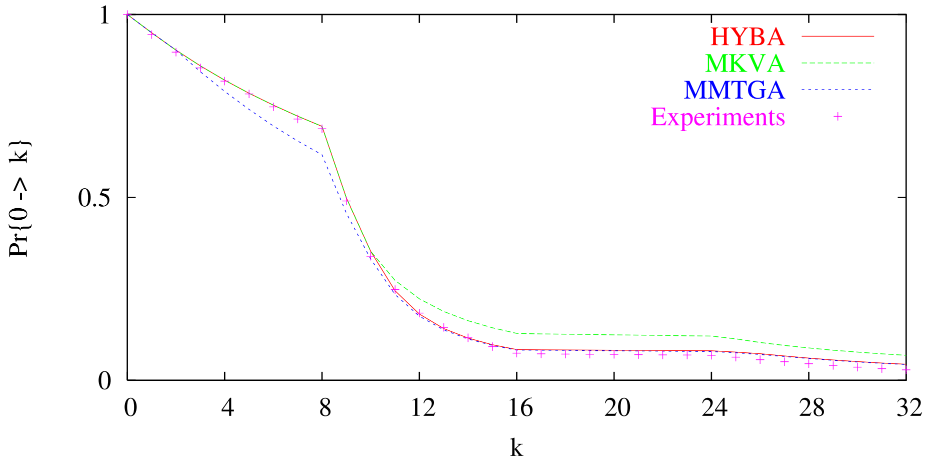


Fig. 5 (*I*) - A. Martini *et al.*, ”A Hybrid Approach for Modeling Stochastic ...”

(*d*)



(*e*)

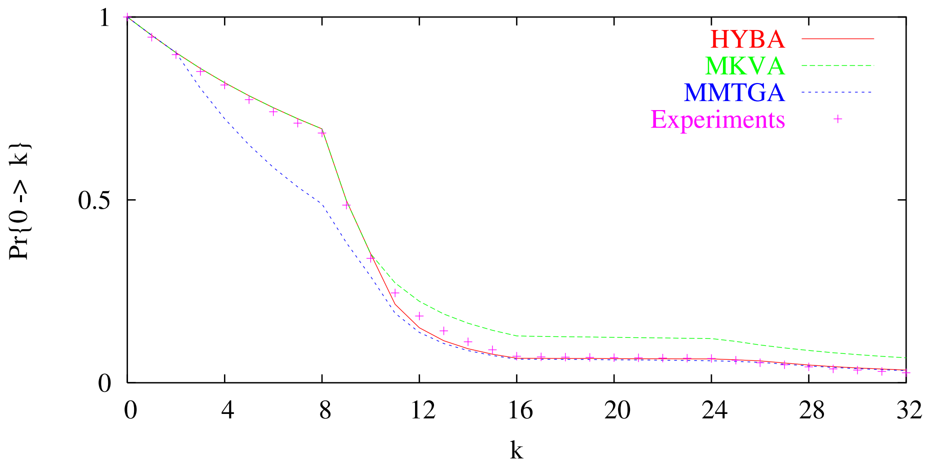


Fig. 5 (*II*) - A. Martini *et al.*, ”A Hybrid Approach for Modeling Stochastic ...”

	HYBA	MKVA	MMTGA
$4L_S$	0.71	0.71	1.55
$4L_D$	0.74	1.42	0.92

Tab. I - A. Martini *et al.*, ”A Hybrid Approach for Modeling Stochastic ...”

θ	HYBA	MKVA	MMTGA
15^o	3.95	3.95	5.34
30^o	2.59	3.20	3.04
45^o	0.55	1.28	0.62
60^o	0.94	3.43	1.72
75^o	0.89	3.66	3.76

Tab. II - A. Martini *et al.*, ”A Hybrid Approach for Modeling Stochastic ...”