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A Percolation Model for the Wave Propagation in Non-Uniform Random Media

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***Abstract* – The statistical characterization of the electromagnetic propagation in urban radio channels is investigated. Towards this end, the propagation medium is modeled with a semi-infinite percolating lattice of lossless scatterers whose density varies in space according to a known one-dimensional distribution. The scenario is illuminated by a monochromatic plane wave that propagates according to the geometrical optics laws. By applying the random processes' theory, a closed form analytical formula is provided, which allows a statistical estimation of the penetration capability of the electromagnetic wave into the medium. The analytical result has been validated through an exhaustive numerical assessment that demonstrates the effectiveness of the proposed model for a wide range of incidence conditions and scatterers' density distribution.**

1 INTRODUCTION

The propagation in urban radio channels is a topic of growing interest due to the rapid spreading of wireless technologies. Because of the high complexity of urban environments, providing an analytical solution by means of the Maxwell's equations is an almost impracticable, or at least very time-expensive, way to solve the problem.

In order to overcome such a drawback, an emerging approach is based on the description of the urban environments with approximate stochastic models to analytically describe the average properties of the electromagnetic propagation. As a matter of fact, there are a large number of books and papers dealing with the electromagnetic propagation in random media by using a wave approach [1]. However, such models assume that the obstacles are either small with respect to the wavelength [2] or large but tenuous compared to the background [3]. Therefore, they are not suitable for the urban scenarios, since scatterers are generally larger than the wavelength and non-tenuous.

An innovative approach was presented in [4], where the urban scenario was described by means of a uniform *percolating lattice* of lossless scatterers, modelling a stochastic distribution of buildings. By assuming simplified propagation mechanisms based on the geometrical optics laws, a closed form solution for the penetration-depth of a plane wave incident on a semi-infinite *percolating lattice* was derived. Successive works (see [5] and the references citied therein) extended the approach to internal-source two dimensional scenarios.

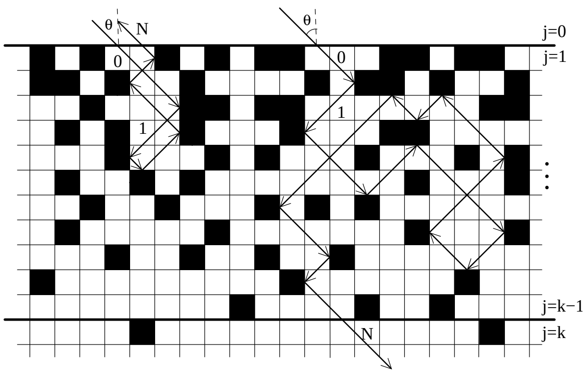

Figure 1: Sketch of the ray propagation in an inhomogeneous percolating lattice.

In the framework of stochastic approaches, this contribution is aimed at extending the approach for the far-external source scenario to the inhomogeneous case, where the density of the obstacles varies with the depth

(Figure 1). This improvement allows a more accurate description of realistic situations as propagation from suburbs to the city centre since the density of the buildings progressively increases.

The paper is organized as follows. In Section 2 the propagation model is presented and a closed-form solution for the wave penetration is provided. Section 3 deals with the numerical validation through selected test cases with different occupation-density profiles and incidence conditions. Final comments and conclusions are drawn in Section 4.

2 MATHEMATICAL FORMULATION

2.1 The percolation model

Let us consider the scenario depicted in Figure 1. A half-plane is partitioned into square cells of unitary length. Each site can be either empty or occupied with a lossless scatterer, with a probability that changes in space with the percolating lattice depth according to a known occupation-density profile

$$\Pr\{\text{cell}\,(j,i)\ \text{occupied}\} = q_j = 1 - p_j, \quad j \geq 1, \qquad (1)$$

where i and j are the indexes of the column and the row of the grid, respectively. According to the percolation theory [6], p_j is greater than the so-called *percolating threshold* p_c ($p_c \approx 0.59275$ for the two-dimensional case).

The source is external to the grid and it radiates a plane monochromatic wave impinging on the lattice with a known angle θ.

By assuming the dimension of the scatterers large compared to the wavelength, the wave propagation is modeled in terms of parallel rays that travel the *lattice* and are reflected according to the geometrical optics laws. Other interactions (i.e., refraction, absorption and diffraction) are neglected.

The propagation of a single ray is described as a realization of the following stochastic process:

$$r_n = r_0 + \sum_{m=1}^{n} x_m, \quad n \geq 0, \quad (2)$$

where r_n is the row where the n-th reflection takes place and

$$x_n = r_n - r_{n-1}, \quad n \geq 1 \qquad (3)$$

models the change of level between n-th and $(n+1)$-th reflections.

For such a scenario, the penetration capability of the electromagnetic wave can be statistically estimated by computing the probability that a ray reaches a level k before being reflected back in the above empty half-plane. Since these events are mutually exclusive, the unknown probability can be expressed as

$$\Pr\{\text{reach level } k\} = \Pr\{r_N \geq k\}, \qquad (4)$$

where N is defined as follows:

$$N = \min\{n : r_n \geq k \text{ or } r_n \leq 0\}. \qquad (5)$$

2.2 Evaluation of the propagation depth

In order to evaluate the propagation depth, it turns out to be useful to express the unknown probability $\Pr\{r_N \geq k\}$ as follows:

$$\Pr\{r_N \geq k\} = \sum_i \Pr\{r_N \geq k \mid r_0 = i\} \times \Pr\{r_0 = i\}, \qquad (6)$$

where $\Pr\{r_0 = i\}$ is the probability mass function of r_0 and $\Pr\{r_N \geq k \mid r_0 = i\}$ represents the probability the ray reaches and eventually goes beyond the level k conditioned to r_0.

2.2.1 Evaluation of $\Pr\{r_N \geq k \mid r_0 = i\}$

The evaluation of $\Pr\{r_N \geq k | r_0 = i\}$ is in order. When $0 < r_0 < k$, it can be demonstrated that the shifted version of the equation (2) with respect to level r_0

$$r_n' = r_n - r_0 = \sum_{m=1}^{n} x_m, \quad n \geq 1, \tag{7}$$

is a *martingale* [7] with respect to the process $\{x_n, n \geq 1\}$. By following the same lines drawn in [4], it can be easily obtained that

$$\Pr\{r_N \geq k | r_0 = i\} \cong \frac{i}{k}. \tag{8}$$

Thus, the final result can be summarized as follows:

$$\Pr\{r_N \geq k | r_0 = i\} \cong \begin{cases} 0, & i = 0 \\ i/k, & 0 < i < k \\ 1, & i \geq k \end{cases} \tag{9}$$

2.2.2 Evaluation of $\Pr\{r_0 = i\}$

As far as $\Pr\{r_0 = i\}$ is concerned, two mutually exclusive situations can be recognized: the ray can be reflected either at the level $i = 0$, without entering the grid, or at a level $i \geq 1$.

In the former case,

$$\Pr\{r_0 = 0\} = q_1, \tag{10}$$

q_1 being the probability a cell is occupied in the first level. In the latter case,

$$\begin{aligned} &\Pr\{r_0 = i\} = \Pr\{\text{to reach level } i\} \times \\ &\Pr\{\text{reflection at level } i | \text{level } i \text{ is reached}\} \end{aligned} \tag{11}$$

It can be demonstrated that the first term of equation (11) is equal to

$$\Pr\{\text{to reach level } i\} = p_1 \prod_{j=1}^{i-1} p_{e_j}^+, \tag{12}$$

where $p_{e_j}^+ = p_j^{\tan\theta} p_{j+1}$ is the probability that a ray, traveling in the positive direction, crosses level j and reaches level $j+1$. On the other hand, the second term of equation (11) can be expressed as

$$\begin{aligned} &\Pr\{\text{reflection at level } i | \text{level } i \text{ is reached}\} \\ &= q_i \sum_{s=0}^{\tan\theta - 1} p_i^s + q_{i+1} p_i^{\tan\theta} = 1 - p_{e_i}^+ = q_{e_i}^+, \end{aligned} \tag{13}$$

$q_{e_j}^+$ being the effective probability that a reflection takes place at level j. Combining equations (10)-(13),

$$\Pr\{r_0 = i\} = \begin{cases} q_1, & i = 0 \\ p_1 q_{e_i}^+ \prod_{j=1}^{i-1} p_{e_i}^+, & i \geq 1 \end{cases} \tag{14}$$

2.2.3 Evaluation of $\Pr\{r_N \geq k\}$

Substituting (9) and (14) in equation (6) yields, after some mathematical manipulations, the following relationship

$$\Pr\{r_N \geq k\} = \sum_{i=1}^{k-1} \frac{i}{k} p_1 q_{e_i}^{+} \prod_{j=1}^{i-1} p_{e_j}^{+} + p_1 \prod_{j=1}^{k-1} p_{e_j}^{+}. \qquad (15)$$

A key-issue must be pointed out. The process $\{r_n^{'}, n \geq 1\}$ can be considered a *martingale* with respect to $\{x_n, n \geq 1\}$ only under the assumption of independent, identically distributed and zero-mean x_n's. By analyzing the probability mass function of x_n, this turns out to be verified when either (*I.a*) $\theta \cong 45^{o}$ or (*I.b*) $n \rightarrow \infty$ and (*II*) if the occupancy profile does not have discontinuities and (*III*) significant variations in the lattice. Moreover, (*IV*) p_j must be neither too close to unit nor too near the percolating threshold [4].

3 NUMERICAL VALIDATION

Many numerical experiments have been carried out in order to assess the effectiveness of the proposed approach as well as its range of validity. Several realistic occupancy profiles and different incidence conditions have been considered. Some representative results are reported in the following.

As a reference, the propagation depth has been estimated by means of computer-based ray tracing experiments [4]. Thus, in order to quantify the accuracy of the approach, the *prediction error*

$$\rho_k \hat{=} \frac{\left|(\Pr\{r_N \geq k\})_P - (\Pr\{r_N \geq k\})_R\right|}{\max_k\left[(\Pr\{r_N \geq k\})_R\right]} \times 100 \qquad (16)$$

and the *mean error*

$$\langle\rho\rangle \hat{=} \frac{1}{M} \sum_{k=1}^{M} \rho_k \qquad (17)$$

have been introduced. In (16) and (17), M is the number of levels of the lattice ($M = 32$). Moreover, the subscripts P and R indicate the value computed by the proposed and by the reference approach, respectively.

The first test case deals with a linear occupancy profile satisfying conditions (*II*), (*III*) and (*IV*),

$$q_j = 0.2 + 3.125 \cdot 10^{-3}\, j. \qquad (18)$$

Figure 2 clearly confirms the optimal fitting between estimated and reference data when $\theta = 45^{o}$ ($\langle\rho\rangle \cong 0.80$). As expected, since condition (*I.a*) is not fully satisfied, the *mean error* increases when θ goes far from 45^{o}. Moreover, since the expected number of reflections increases as θ grows (*I.b*), the *mean error* turns out to be lower for the angle values greater than $\theta = 45^{o}$ ($\langle\rho\rangle_{\theta=30^{o}} / \langle\rho\rangle_{\theta=60^{o}} \cong 2.3$ and $\langle\rho\rangle_{\theta=15^{o}} / \langle\rho\rangle_{\theta=75^{o}} \cong 1.75$).

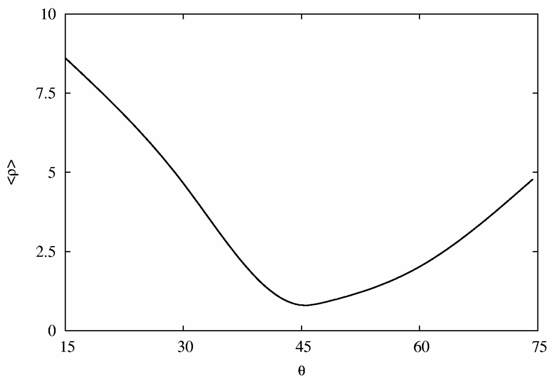

Figure 2: Linear occupancy profile – *Mean error* versus the value of the incidence angle.

In the second test case, two exponential-like occupancy profiles satisfying condition (*II*) and (*IV*) are considered,

$$q_j = \begin{cases} \alpha \cdot \exp[(j-L)\cdot\tau] & j \le L \\ \alpha \cdot \exp[(L-j)\cdot\tau] & j > L \end{cases}, \qquad (19)$$

where L is equal to $M/2$ and the parameters α and τ are given in Table 1 for each profiles.

	α	τ
double exp 1	0.3	25.34×10^{-3}
double exp 2	0.3	68.66×10^{-3}

Table 1: Double-exponential occupancy probability – Descriptive parameters.

By assuming that condition (*I*) holds true (i.e., $\theta = 45^o$), Figure 3 shows that, whatever k, the proposed approach satisfactorily performs when the "*double exp 1*" profile is considered ($\langle\rho\rangle \cong 1.26$). Instead, the *mean error* increases when the "*double exp 2*" distribution is taken into account (see Fig. 4) since condition (*III*) loses validity.

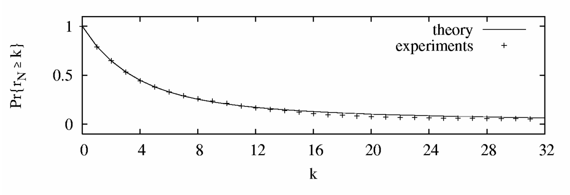

Figure 3: Double-exponential occupancy probability – Profile "*double exp 1*" – Estimated (solid line) and reference (crosses) values of $\Pr\{r_N \ge k\}$

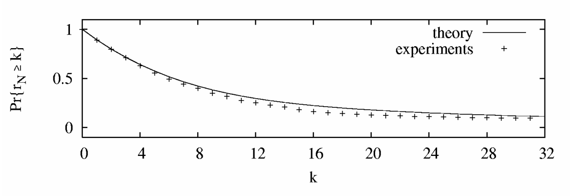

Figure 4: Double-exponential occupancy probability – Profile "*double exp 2*" – Estimated (solid line) and reference (crosses) values of $\Pr\{r_N \ge k\}$

4 CONCLUSIONS

The statistical characterization of the electromagnetic propagation has been considered. In such a framework, an extension of the approach presented in [4] to non-uniform random lattices has been presented. Numerical experiments have confirmed the effectiveness of the proposed analytical approach.

References

[1] A. Ishimaru, "Wave Propagation and Scattering in Random Media", Academic Press, 1978.

[2] M. Kerker, "The scattering of Light", Academic Press, New York, 1969.

[3] B. J. Berne and R, Pecora, "Dynamic Light Scattering", J. Wiley, New York, 1976.

[4] G. Franceschetti et al., "Propagation without wave equation towards and urban area model", IEEE Trans. Antennas Propag., **AP-47**, 1393-1404, 1999.

[5] S. Marano and M. Franceschetti, "Ray propagation in a random lattice, a maximum entropy, anomalous diffusion process", internal report DISI.

[6] G. Grimmet, "Percolation", Springer-Verlag, New York, 1989.

[7] R. M. Ross, "Stochastic Processes", J. Wiley, New York, 1983.