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P. Rocca, M. Donelli, G.L. Gragnani, and A. Massa

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Paolo Rocca,* Massimo Donelli,* *Member*, *IEEE*, Gian Luigi Gragnani,** and Andrea Massa* *Member*, *IEEE*,

* *ELEDIA* Research Group

Department of Information and Communication Technologies - University of Trento

Via Sommarive 14, 38050 Trento - Italy

Tel. +39 0461 882057, Fax +39 0461 882093

E-mail: *andrea.massa@ing.unitn.it,*

{*paolo.rocca, massimo.donelli*}@*disi.unitn.it*

** Department of Biophysical and Electronic Engineering - University of Genoa,

Via Opera Pia 11/A, 16145 Genoa, Italy

Tel. +39 010 3532244, Fax +39 010 3532245

E-mail: *gragnani@dibe.unige.it*

Some Indications on the Analytic Expression of Non-Radiating Currents in 2D TM Imaging Problems

Paolo Rocca*, Massimo Donelli*, Gian Luigi Gragnani**, and Andrea Massa*

Abstract

In this contribution, an analytical procedure for building the non-radiating part of the equivalent current in inverse scattering problems is presented. The mathematical formulation, concerned with a $2D$ configuration and TM illumination, is provided and discussed.

Key words:

Inverse scattering, Non-radiating currents.

1 Introduction

In the framework of inverse scattering techniques, several techniques consider the introduction of an equivalent current density defined into the dielectric domain. Accordingly, the problem is formulated in terms of an "inverse source" one through the introduction of an equivalent current in order to linearize the original inverse scattering problem [1][2][3][4]. Unfortunately, a typical drawback of inverse source problems relies in their non-uniqueness due to the presence of non-radiating components of the equivalent currents. As far as the presence of non-radiating currents in a volume formulation is concerned, the issue has been discussed in [5] and an approach is presented for the reconstruction of the non-radiating part of the equivalent volumetric source. Moreover, in [6], a wide review and some representative results on the presence of non-radiating currents as well as about the relationship between non-radiating sources and scattering objects that could be invisible under particular illumination conditions are reported. As a matter of fact, although it has been proved in [7] that a perfectly invisible object does not exist, on the other hand, it has been also demonstrated that a scattering object can be nonscattering at particular directions of illumination [8], thus the equivalent inverse source problem include non-radiating components (i.e. the null space of the operator is not empty). In the following, a preliminary procedure for building a non-radiating current is reported.

2 Mathematical Formulation

In the framework of inverse source problems, non-radiating currents, $Q_{NR}(\bar{r})$, are currents that "produce fields that are identically zero outside their source volume" [9], hence they satisfy the following integral relationship

$$\int\int\int_V Q_{NR}(\bar{r}')G_0(|\bar{r}-\bar{r}'|)d\bar{r}' \equiv 0 \quad \bar{r}' \in V,\, \bar{r} \notin V \tag{1}$$

where the $G_0(|\bar{r}-\bar{r}'|)$ is the free-space Green's function. In the following, let us consider a current $Q(\bar{r})$ whose support is a two dimensional domain D. As far as $2D$ problems with TM illuminations are concerned, $Q(\bar{r})$ is non radiating [$Q(\bar{r}) = Q_{NR}(\bar{r})$] if

$$f(\bar{\rho}) = \int_D J_0(k_0|\bar{\rho}-\bar{\rho}'|)Q(\bar{\rho}')d\bar{\rho}' \equiv 0\ , \quad \forall\, \bar{\rho} \tag{2}$$

$J_0(x)$ being the (cylindrical) Bessel function of the first kind and zero-th order and $|\bar{\rho}| = \rho = \sqrt{x^2+y^2}$.

2.1 Building Non-Radiating Currents

It is well known that a set of non radiating currents can be defined for a spherical domain in $3D$ case [6][10]. Concerning the $2D$ TM case, let us consider a current whose support is a circular domain of radius a, and having the following expression:

$$Q(\bar{\rho}) = Q(\rho, \vartheta) = [\alpha_1 J_n(l_1 \rho / a) + \alpha_2 J_n(l_2 \rho / a)]\, e^{jn\vartheta} \tag{3}$$

where l_1 and l_2 are two different zeroes of $J_n(x)$ and α_1 and α_2 are constants. In order to analyze the non-radiating condition (2), let us compute $f(\bar{\rho})$

$$f(\bar{\rho}) = \int_D J_0(k_0 |\bar{\rho} - \bar{\rho}'|) Q(\bar{\rho}') d\bar{\rho}' \quad D = \{\bar{\rho} \,:\, \rho \leq a\} \tag{4}$$

Towards this end, let us consider that

$$J_0(k_0 |\bar{\rho} - \bar{\rho}'|) = \sum_{\mu=-\infty}^{+\infty} J_\mu(k\rho') J_\mu(k\rho) e^{j\mu\vartheta} e^{-j\mu\vartheta'} \tag{5}$$

and since $d\bar{\rho}' = \rho' d\rho' d\vartheta'$, by substituting (5) in (4), we get

$$\begin{gathered}\int_{\rho \leqslant a} J_0(k_0 |\bar{\rho} - \bar{\rho}'|) Q(\bar{\rho}') d\bar{\rho}' = \\ = \int_0^{2\pi} \int_0^a \left[\sum_{\mu=-\infty}^{+\infty} J_\mu(k\rho') J_\mu(k\rho) e^{j\mu\vartheta} e^{-j\mu\vartheta'}\right] [\alpha_1 J_n(l_1 \rho'/a) + \alpha_2 J_n(l_2 \rho'/a)]\, e^{jn\vartheta'} \rho' d\rho' d\vartheta' = \\ = \int_0^{2\pi} \int_0^a J_n(k\rho') J_n(k\rho) e^{j\mu\vartheta} [\alpha_1 J_n(l_1 \rho'/a) + \alpha_2 J_n(l_2 \rho'/a)]\, \rho' d\rho' d\vartheta' = \\ = 2\pi J_n(k\rho) e^{j\mu\vartheta} \left[\alpha_1 \int_0^a J_n(k\rho') J_n(l_1 \rho'/a) \rho' d\rho' + \alpha_2 \int_0^a J_n(k\rho') J_n(l_2 \rho'/a) \rho' d\rho'\right]\end{gathered} \tag{6}$$

Moreover, let us recall that

$$\int J_\mu(b_1 x) J_\mu(b_2 x) x dx = \frac{x}{b_1^2 - b_2^2} [b_1 J_{\mu+1}(b_1 x) J_\mu(b_2 x) - b_2 J_\mu(b_1 x) J_{\mu+1}(b_2 x)] + \boldsymbol{C} \tag{7}$$

hence

$$\alpha_1 \int_0^a J_n(k\rho') J_n(l_1 \rho'/a) \rho' d\rho' = -\alpha_1 \frac{a^2}{a^2 k^2 - l_1^2} l_1 J_n(ka) J_{n+1}(l_1) \tag{8}$$

and

$$\alpha_2 \int_0^a J_n(k\rho') J_n(l_2 \rho'/a) \rho' d\rho' = -\alpha_2 \frac{a^2}{a^2 k^2 - l_2^2} l_2 J_n(ka) J_{n+1}(l_2)\,. \tag{9}$$

By rearranging all terms, it turns out that

$$f(\bar{\rho}) = -2\pi J_n(k\rho)e^{j\mu\vartheta}\left[\alpha_1\frac{a^2}{a^2k^2-l_1^2}l_1J_n(ka)J_{n+1}(l_1)+\alpha_2\frac{a^2}{a^2k^2-l_2^2}l_2J_n(ka)J_{n+1}(l_2)\right]; \tag{10}$$

According to (2), $Q(\bar{\rho}) = Q_{NR}(\bar{\rho})$ if $f(\bar{\rho}) = 0$, therefore since the following condition

$$\alpha_2 = -\alpha_1\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)} \tag{11}$$

nulls (10), then a non-radiating current is obtained by substituting (11) in (3), thus obtaining the following expression

$$Q_{NR}(\overline{\rho}) = \alpha_1\left[J_n(l_1\rho/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho/a)\right]e^{jn\vartheta}. \tag{12}$$

2.2 Testing the Non-Radiating Condition

Let us consider a $2D$ TM problem where a cylindrical scatterer of circular cross-section of radius a lies in free-space. Our aim is to verify whether (1) is satisfied when considering $Q_{NR}(\overline{\rho})$ defined in (12).

In such a scenario, the integral in (1) turns out to be equal to

$$\int\int_{\rho\leqslant a} H_0^{(2)}(k_0|\bar{\rho}-\bar{\rho}'|)Q(\bar{\rho}')d\bar{\rho}' \tag{13}$$

since $G_0(|\bar{\rho}-\bar{\rho}'|) = H_0^{(2)}\left(k_0\left|\bar{\rho}-\bar{\rho}'\right|\right)$, $H_0^{(2)}$ being the Hankel function of the second kind of order zero. Thus, we should verify that

$$\int\int_{\rho\leqslant a} H_0^{(2)}(k_0|\bar{\rho}-\bar{\rho}'|)Q(\bar{\rho}')d\bar{\rho}' = \begin{cases} 0 & \rho > a \\ \neq 0 & \rho \leqslant a \end{cases} \tag{14}$$

with $Q(\bar{\rho}) = Q_{NR}(\bar{\rho}) = \alpha_1\left[J_n(l_1\rho/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho/a)\right]e^{jn\vartheta}$;

Towards this end, let us consider the following relationships

$$H_0^{(2)}\left(k\left|\bar{\rho}-\bar{\rho}'\right|\right) = \sum_{\mu=-\infty}^{+\infty} J_\mu(k\rho')H_\mu^{(2)}(k\rho)\mathrm{e}^{j\mu(\vartheta-\vartheta')} \qquad \rho \geqslant \rho' \tag{15}$$

$$H_0^{(2)}\left(k\left|\bar{\rho}-\bar{\rho}'\right|\right) = \sum_{\mu=-\infty}^{+\infty} J_\mu(k\rho)H_\mu^{(2)}(k\rho')\mathrm{e}^{j\mu(\vartheta-\vartheta')} \qquad \rho \leqslant \rho' \tag{16}$$

and let us first substitute (15) and then (16) into equation (14).

In particular, when $\rho > a$

$$
\begin{aligned}
&\int\int_{\rho\leqslant a}\left[\sum_{\mu=-\infty}^{+\infty} J_\mu(k\rho')H_\mu^{(2)}(k\rho)\mathrm{e}^{j\mu(\vartheta-\vartheta')}\right]\alpha_1\left[J_n(l_1\rho'/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho'/a)\right]\mathrm{e}^{jn\vartheta'}\rho'\mathrm{d}\rho'\mathrm{d}\vartheta' =\\
&= \int_0^{2\pi}\int_0^a\left[J_n(k\rho')H_n^{(2)}(k\rho)\mathrm{e}^{jn\vartheta}\right]\alpha_1\left[J_n(l_1\rho'/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho'/a)\right]\rho'\mathrm{d}\rho'\mathrm{d}\vartheta' =\\
&= H_n^{(2)}(k\rho)\mathrm{e}^{jn\vartheta}\int_0^{2\pi}\mathrm{d}\vartheta'\int_0^a J_n(k\rho')\alpha_1\left[J_n(l_1\rho'/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho'/a)\right]\rho'\mathrm{d}\rho' =\\
&= 2\pi H_n^{(2)}(k\rho)\mathrm{e}^{jn\vartheta}\int_0^a J_n(k\rho')\alpha_1\left[J_n(l_1\rho'/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho'/a)\right]\rho'\mathrm{d}\rho' =\\
&= 2\pi H_n^{(2)}(k\rho)\mathrm{e}^{jn\vartheta}\left[\alpha_1\int_0^a J_n(k\rho')J_n(l_1\rho'/a)\rho'\mathrm{d}\rho'-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}\int_0^a J_n(k\rho')J_n(l_2\rho'/a)\rho'\mathrm{d}\rho'\right] = 0
\end{aligned}
\tag{17}
$$

Since the two integrals within the square bracket in the last row of (17) are the same of Eq. (6), where their sum has been made null by imposing condition (11), it has been proved that the field radiated by a non-radiating current is null outside its support.

Otherwise, the field inside the support of the current ($\rho \le a$) is given by

$$
\begin{aligned}
&\int\int_{\rho\leqslant a}\left[\sum_{\mu=-\infty}^{+\infty} J_\mu(k\rho)H_\mu^{(2)}(k\rho')\mathrm{e}^{j\mu(\vartheta-\vartheta')}\right]\alpha_1\left[J_n(l_1\rho'/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho'/a)\right]\mathrm{e}^{jn\vartheta'}\rho'\mathrm{d}\rho'\mathrm{d}\vartheta' =\\
&= \int_0^{2\pi}\int_0^a\left[J_n(k\rho)H_n^{(2)}(k\rho')\mathrm{e}^{jn\vartheta}\right]\alpha_1\left[J_n(l_1\rho'/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho'/a)\right]\rho'\mathrm{d}\rho'\mathrm{d}\vartheta' =\\
&= 2\pi J_n(k\rho)\mathrm{e}^{jn\vartheta}\int_0^a H_n^{(2)}(k\rho')\alpha_1\left[J_n(l_1\rho'/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho'/a)\right]\rho'\mathrm{d}\rho'
\end{aligned}
\tag{18}
$$

Since

$$H_n^{(2)}(x) = J_n(x) - jY_n(x) \tag{19}$$

where $J_n(x)$ and $Y_n(x)$ are the Bessel functions of the first and second kind of order n, respectively, by substituting (19) into (18) it results that

$$
\begin{aligned}
&2\pi J_n(k\rho)\mathrm{e}^{jn\vartheta}\int_0^a\left[J_n(k\rho')-jY_n(k\rho')\right]\alpha_1\left[J_n(l_1\rho'/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho'/a)\right]\rho'\mathrm{d}\rho' =\\
&-j2\pi J_n(k\rho)\mathrm{e}^{jn\vartheta}\int_0^a Y_n(k\rho')\alpha_1\left[J_n(l_1\rho'/a)-\frac{a^2k^2-l_2^2}{a^2k^2-l_1^2}\frac{l_1}{l_2}\frac{J_{n+1}(l_1)}{J_{n+1}(l_2)}J_n(l_2\rho'/a)\right]\rho'\mathrm{d}\rho' \neq 0
\end{aligned}
$$

where for the real part the same conclusions as obtained in (17) holds true, while the imaginary part is not negligible and generally it is different to zero, thus demonstrating that the scattered field inside the support of the current is not necessarily null. Hence a current defined as in (12) in a $2D$ TM problem does not radiate outside its support, but it provides a field that is not null inside the source domain. This is a non-radiating current.

3 Conclusions

In this paper, a procedure for indicating the non-radiating part of the equivalent current in inverse scattering problems when formulated in terms of an inverse source and concerned with a $2D$ configuration and TM illumination, is presented.

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